

TAKÁCS' ASYMPTOTIC THEOREM AND ITS APPLICATIONS: A SURVEY

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ABSTRACT. The book of Lajos Takács *Combinatorial Methods in the Theory of Stochastic Processes* has been published in 1967. It discusses various problems associated with

$$(*) \quad P_{k,i} = P \left\{ \sup_{1 \leq n \leq \rho(i)} (N_n - n) < k - i \right\},$$

where $N_n = \nu_1 + \nu_2 + \dots + \nu_n$ is a sum of mutually independent, nonnegative integer and identically distributed random variables, $\pi_j = P\{\nu_k = j\}$, $j \geq 0$, $\pi_0 > 0$, and $\rho(i)$ is the smallest n such that $N_n = n - i$, $i \geq 1$. (If there is no such n , then $\rho(i) = \infty$.)

(*) is a discrete generalization of the classic ruin probability, and its value is represented as $P_{k,i} = Q_{k-i}/Q_k$, where the sequence $\{Q_k\}_{k \geq 0}$ satisfies the recurrence relation of convolution type: $Q_0 \neq 0$ and $Q_k = \sum_{j=0}^k \pi_j Q_{k-j+1}$.

Since 1967 there have been many papers related to applications of the generalized classic ruin probability. The present survey concerns only with one of the areas of application associated with asymptotic behavior of Q_k as $k \rightarrow \infty$. The theorem on asymptotic behavior of Q_k as $k \rightarrow \infty$ and further properties of that limiting sequence are given on pages 22-23 of the aforementioned book by Takács. In the present survey we discuss applications of Takács' asymptotic theorem and other related results in queueing theory, telecommunication systems and dams. Many of the results presented in this survey have appeared recently, and some of them are new. In addition, further applications of Takács' theorem are discussed.

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1. INTRODUCTION

The book of Takács [75] has been published in 1967. It discusses various problems associated with

$$(1.1) \quad P_{k,i} = P \left\{ \sup_{1 \leq n \leq \rho(i)} (N_n - n) < k - i \right\},$$

where $N_n = \nu_1 + \nu_2 + \dots + \nu_n$ is a sum of mutually independent, nonnegative integer and identically distributed random variables, $\pi_j = P\{\nu_k = j\}$, $j \geq 0$, $\pi_0 > 0$, and $\rho(i)$ is the smallest n such that $N_n = n - i$, $i \geq 1$. (If there is no such n , then $\rho(i) = \infty$.)

(1.1) is a discrete generalization of classic ruin probability, and its value is represented as $P_{k,i} = \frac{Q_{k-i}}{Q_k}$, where the sequence $\{Q_k\}_{k \geq 0}$ satisfies the recurrence relation of convolution type:

$$(1.2) \quad Q_k = \sum_{j=0}^k \pi_j Q_{k-j+1},$$

with arbitrary $Q_0 \neq 0$ (see Theorem 2 [75] on page 18). The probability generating function of Q_n is

$$(1.3) \quad Q(z) = \sum_{k=0}^{\infty} Q_k z^k = \frac{Q_0 \pi(z)}{\pi(z) - z},$$

where $\pi(z) = \sum_{k=0}^{\infty} \pi_k z^k$.

There is also a continuous generalization of the classical ruin theorem in [75].

Different applications of the aforementioned recurrence relation have been provided for random walks, Brownian motion, queueing processes, dam and storage processes, risk processes and order statistics in [75]. Most of these applications in [75] and following papers [76] – [84] are associated with an explicit application of a generalization of the ruin probability formula or its continuous analogue.

The ruin problems and ballot theorems have a long history and attract wide attention in the literature. The solution of the first problems related to this subject had been given in 1887 due to Bertrand [23] (see Sheinin [69] for the review of Bertrand's work). For other papers related to ballots problems, their generalizations and applications see [46], [47], [49], [54], [55], [57], [58], [82], [85], [87] and many others.

The present survey is concerned with some special application of recurrence relation (1.2), where there is no explicit application of the generalized gambler ruin problem. In our application it does not matter what the meaning of values π_j is, and what then the fraction $\frac{Q_{k-i}}{Q_k}$ means.

In this survey we discuss applications of asymptotic Theorem 5 of Takács [75] formulated on page 22 and the further asymptotic results on page 23 associated with this theorem. We also use the other asymptotic results (Tauberian theorems of Postnikov which will be mentioned later). These asymptotic results develop the aforementioned asymptotic theorem of Takács [75] and are used in the paper together with Takács' theorem to accomplish it. Nevertheless, the title of this paper is supported by the fact that Takács' theorem serves as a tool to establish the main asymptotic properties of processes, while Postnikov's Tauberian theorems improve and strengthen the results on that asymptotic behavior, when additional conditions are satisfied. In some delicate cases, applications of Postnikov's Tauberian theorems have especial significance, when the case of Takács' theorem cannot help to establish a required property.

For our convenience, we use the variant of Theorem 5 of [75] in combination with formula (35) on page 23 as follows. Denote

$$\gamma_\ell = \sum_{j=\ell}^{\infty} \prod_{k=1}^{\ell} (j - k + 1) \pi_j.$$

The value γ_ℓ characterizes the ℓ th factorial moment related to the probabilities π_j , $j \geq 0$.

Theorem 1.1. *Let $\pi_0 > 0$. If $\gamma_1 < 1$, then*

$$(1.4) \quad \lim_{k \rightarrow \infty} Q_k = \frac{Q_0}{1 - \gamma_1}.$$

If $\gamma_1 = 1$ and $\gamma_2 < \infty$, then

$$(1.5) \quad \lim_{k \rightarrow \infty} \frac{Q_k}{k} = \frac{2Q_0}{\gamma_2}.$$

If $\gamma_1 > 1$, then

$$(1.6) \quad \lim_{k \rightarrow \infty} \left[Q_k - \frac{Q_0}{\sigma^k(1 - \pi'(\sigma))} \right] = \frac{Q_0}{1 - \gamma_1},$$

where σ is the least nonnegative root of the equation $z = \pi(z)$, and $0 < \sigma < 1$.

In the case $\gamma_2 = \infty$, [75] recommends to use the Tauberian theorem of Hardy and Littlewood [44], [43], which states that if

$$Q(z) \asymp \frac{1}{(1-z)^{\alpha+1}} L\left(\frac{1}{1-z}\right),$$

as $z \uparrow 1$, where $\alpha \geq 0$ and $L(x)$ is a slowly varying function at infinity (i.e. $L(cx) \asymp L(x)$ for any positive c as $x \rightarrow \infty$), then

$$Q_k \asymp \frac{k^\alpha}{\Gamma(\alpha+1)} L(k),$$

as $k \rightarrow \infty$, where $\Gamma(x)$ is the Euler gamma-function.

The case $\gamma_1 = 1$ and $\gamma_2 < \infty$ has been developed by Postnikov [65], Sect. 25. He established the following two Tauberian theorems.

Theorem 1.2. *Let $\pi_0 > 0$. Suppose that $\gamma_1 = 1$ and $\gamma_3 < \infty$. Then, as $k \rightarrow \infty$,*

$$(1.7) \quad Q_k = \frac{2Q_0}{\gamma_2} n + O(\log n).$$

Theorem 1.3. *Let $\pi_0 > 0$. Suppose that $\gamma_1 = 1$, $\gamma_2 < \infty$ and $\pi_0 + \pi_1 < 1$. Then, as $k \rightarrow \infty$,*

$$(1.8) \quad Q_{k+1} - Q_k = \frac{2Q_0}{\gamma_2} + o(1).$$

The applications reviewed in this survey are related to queueing theory, telecommunication systems and dams/storage systems.

The first paper in this area has been published in 1981 by the author [3]. The asymptotic behavior of the mean busy period of the $M/G/1$ queueing system with service depending on queue-length has been studied there. This and the several later papers have been summarized in book [5]. Other research papers, associated with application of (1.2) have appeared recently [7], [9] – [13]. In this survey we also discuss many other closely related papers, the results of which have been obtained by many authors. Several new results related to the asymptotic behavior of consecutive losses in $M/GI/1/n$ queueing systems are also established (Sect. 3), and they then applied for analysis of consecutive losses in telecommunication systems (Sect. 8).

The survey is organized as follows. In Sect. 2-6, which are the first part of this survey, we discuss the classical loss queueing systems such as $M/GI/1/n$, $GI/M/1/n$ and $GI/M/m/n$. For these systems we discuss applications of Takács' theorem 1.1 and Postnikov's theorems 1.2 and 1.3 for analysis of the limiting stationary loss probabilities. Sect. 3 contains new results on consecutive losses. In Sect. 6 we discuss further research problems related to asymptotic analysis of large retrial queueing systems.

In Sect. 7-12, which, respectively, are the second part of this paper, we discuss various applications of these theorems for different models of telecommunication systems and dams/storage systems. All of these applications of Takács' theorem are not traditional, and, being specific, the models considered here describe real-world systems or models for these systems. The results of Sect. 8 are new. They are based on applications of new results on consecutive losses, presented in Sect. 3. In Sect. 11 we discuss further research problems and the ways of their solution associated with asymptotic analysis and optimal control of large dams presented in Sect. 9 and 10.

Part 1. Applications to queueing systems

2. LOSSES IN THE $M/GI/1/n$ QUEUEING SYSTEM

Considering $M/GI/1/n$ queue, Tomkó [88] was the first person to establish recurrence relation (1.2) for the expected busy periods. (We assume that n is the number of waiting places excluding the service space.) Let T_n denote a busy period, and let T_k , $k = 0, 1, \dots, n-1$, be busy periods associated with similar queueing systems (i.e. having the same rate λ of Poisson input and the same probability distribution function $B(x)$ of a service time) but only having different number of

waiting places. Namely Tomkó [88] has derived

$$(2.1) \quad ET_n = \sum_{j=0}^n E(T_{n-j+1}) \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x).$$

In the case of deterministic service times, Tomkó [88] used Karamata's Tauberian theorem [89], Chapt. 5.

Representation (2.1) has been also obtained by Cohen [32], Cooper and Tilt [33], Rosenlund [67] as well as by the author in [3] and [5].

The relationship between the mean busy period in the $M/GI/1/n$ queueing system and the maximum queue-length distribution in the $M/GI/1$ queueing system has been established by Cooper and Tilt [33]. Namely, they derived Takács' result [78] on the supremum queue-length distribution of the $M/G/1$ queue during a busy period by the methods distinguished from the original Takács' method.

With the aid of Theorems 1.1, 1.2 and 1.3 one can easily study the asymptotic behavior of ET_n as $n \rightarrow \infty$. Immediately from these theorems we correspondingly have as follows.

Theorem 2.1. *If $\rho < 1$, then*

$$\lim_{n \rightarrow \infty} ET_n = \frac{\rho}{\lambda(1 - \rho)}.$$

If $\rho = 1$ and $\rho_2 = \lambda^2 \int_0^\infty x^2 dB(x) < \infty$, then

$$\lim_{n \rightarrow \infty} \frac{ET_n}{n} = \frac{2}{\lambda \rho_2}.$$

If $\rho > 1$, then

$$(2.2) \quad \lim_{n \rightarrow \infty} \left\{ ET_n - \frac{\rho}{\lambda[1 + \lambda \widehat{B}'(\lambda - \lambda\varphi)]\varphi^n} \right\} = \frac{\rho}{\lambda(1 - \rho)},$$

where $\widehat{B}(s) = \int_0^\infty e^{-sx} dB(x)$ is the Laplace-Stieltjes transform of a service time, and φ is the least positive root of the equation

$$(2.3) \quad z = \widehat{B}(\lambda - \lambda z).$$

Theorem 2.2. *If $\rho = 1$ and $\int_0^\infty x^3 dB(x) < \infty$, then as $n \rightarrow \infty$*

$$ET_n = \frac{2}{\lambda \rho_2} n + O(\log n).$$

Theorem 2.3. *If $\rho = 1$ and $\rho_2 < \infty$, then*

$$ET_n - ET_{n-1} = \frac{2}{\lambda \rho_2} + o(1).$$

The proof of Theorems 2.1 and 2.2 follows immediately from the statements of Theorems 1.1 and 1.2 correspondingly. Let us prove Theorem 2.3.

Proof. To prove this theorem, it is enough to show that for all $\lambda > 0$

$$(2.4) \quad \widehat{B}(\lambda) - \lambda \widehat{B}'(\lambda) < 1.$$

Then all of the conditions of Theorem 1.3 will be satisfied, and the statement of this theorem will follow immediately from Theorem 1.3.

Let us prove (2.4). Taking into account that

$$\sum_{j=0}^{\infty} \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x) = \int_0^{\infty} \sum_{j=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x) = 1,$$

and

$$\int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x) \geq 0$$

we have

$$\widehat{B}(\lambda) - \lambda \widehat{B}'(\lambda) \leq 1.$$

Therefore (2.4) will be proved if we show that for some λ_0 the equality

$$(2.5) \quad \widehat{B}(\lambda_0) - \lambda_0 \widehat{B}'(\lambda_0) = 1$$

is not the case. Indeed, since $\widehat{B}(\lambda) - \lambda \widehat{B}'(\lambda)$ is an analytic function in λ , then, according to the theorem on maximum absolute value of an analytic function, equality (2.5) is to valid for all $\lambda > 0$, i.e. $\widehat{B}(\lambda) - \lambda \widehat{B}'(\lambda) = 1$. This means that (2.5) is valid if and only if $\int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x) = 0$ for all $j \geq 2$ and for all $\lambda > 0$. This in turn implies that $\widehat{B}(\lambda)$ must be a linear function, i.e. $\widehat{B}(\lambda) = c_0 + c_1 \lambda$ for some constants c_0 and c_1 . However, because of $|\widehat{B}(\lambda)| \leq 1$, we obtain $c_0 = 1$ and $c_1 = 0$. This corresponds to the trivial case, where the probability distribution function $B(x)$ is concentrated in point 0. This case cannot be valid, since λ is assumed to be strictly positive. Therefore (2.5) is not the case, and hence (2.4) holds. \square

Let L_n denote the number of losses during the busy period T_n , and let ν_n denote the number of served customers during the same busy period.

Using Wald's equations [41], p.384, we have the following system.

$$(2.6) \quad E\nu_n + EL_n = \lambda ET_n + 1,$$

$$(2.7) \quad E\nu_n = \mu ET_n,$$

where μ denotes the reciprocal of the expected service time. The term λET_n of the right-hand side of (2.6) denotes the expected number of arrivals during a regeneration period excluding the first (tagged) customer, who starts the busy period. Equations (2.6) and (2.7) are similar to those (1) and (2) of paper [8].

From these two equations (2.6) and (2.7), by subtracting the second from the first, we obtain:

$$EL_n = (\lambda - \mu)ET_n + 1,$$

leading to

$$(2.8) \quad EL_n - 1 = (\lambda - \mu)ET_n.$$

Equation (2.8) coincides with (7) of [7] which is written under another notation and was derived by the different method. From (2.8) we obtain the following recurrence relation:

$$(2.9) \quad EL_n - 1 = \sum_{j=0}^n (EL_{n-j+1} - 1) \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x), \quad EL_0 = \frac{\lambda}{\mu} = \rho.$$

Recurrence relation (2.9) is the same as (3) of [7] and enables us to prove the following theorem (see [4], [5] and [7]).

Theorem 2.4. *If $\rho < 1$, then*

$$\lim_{n \rightarrow \infty} \text{EL}_n = 0.$$

If $\rho = 1$, then for all $n \geq 0$

$$(2.10) \quad \text{EL}_n = 1.$$

If $\rho > 1$, then

$$(2.11) \quad \lim_{n \rightarrow \infty} \left[\text{EL}_n - \frac{(\rho - 1)\varphi^{-n}}{1 + \lambda \widehat{B}'(\lambda - \lambda\varphi)} \right] = 0,$$

where $\widehat{B}(s) = \int_0^\infty e^{-sx} dB(x)$ is the Laplace-Stieltjes transform of a service time, and φ is the least positive root of the equation (2.3).

Evidently, that the statement of this theorem follows immediately by application of Theorem 1.1. In this case we equate π_j with

$$\int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB(x),$$

and the condition $\pi_0 > 0$ is equivalent to $\widehat{B}(\lambda) > 0$ and is satisfied. Then we equate γ_1 with ρ and Q_k with $\text{EL}_k - 1$ and easily arrive at the statement of the theorem, where under the condition $\rho > 1$, there is a unique root of equation (2.3) belonging to the interval $(0, 1)$ (e.g. Takács [74]).

The most significant consequence of this theorem is (2.10). This result met attention in the literature, and there is a number of papers that devoted to extension of this result. We mention [8], [63], [66], [64], [95] as well as [14]. On the other hand, rel. (2.10) is an unexpected generalization of the following elementary result from random walk theory¹. (This random walk is associated with the Markovian $M/M/1/n$ queueing system.) Namely, let X_0, X_1, \dots be random variables taking the values ± 1 with equal probability $\frac{1}{2}$. Let $S_0 = 0$. Then $S_n = X_1 + X_2 + \dots + X_n$ characterizes a symmetric random walk starting at zero, and $\tau = \inf\{l > 0 : S_l = 0\}$ is the moment of the first return to zero point. Then for any level $m \neq 0$, the expectation of the number of up-crossings across the level m during the random time $[0, \tau]$ is equal to $\frac{1}{2}$.

For $\rho > 1$, the asymptotic analysis of ET_n and EL_n of the $M/GI/1/n$ queueing system has also been provided in Azlarov and Takhirov [18] and Azlarov and Tashmanov [19] by methods of the complex analysis. The similar results for the $GI/M/1/n$ queue have been obtained by Takhirov [86]. The asymptotic estimations obtained in these papers have the worse order of the remainder than that following from Takács' theorem 1.1.

Theorem 2.4 has been extended in [6] for the case of the $M/GI/1$ queueing system where if an arriving customer meets n or more customers in the queue, then he joins the queue with probability p or lost with probability $q = 1 - p$.

The case $\rho = 1 + \delta$ where $\delta n \rightarrow C \geq 0$ as $\delta \rightarrow 0$ and $n \rightarrow \infty$ falls into the area of the heavy traffic analysis. Under this assumption we have the following theorem (see, Theorem 2 of [7]).

¹The result is not widely known. The mention about this result can be found, for example, in the book of Szekely [72]. For another relevant consideration see Wolff [94], p. 411

Theorem 2.5. *Let $\rho = 1 + \delta$, $\delta > 0$. Denote $\rho_j(\delta) = \int_0^\infty (\lambda x)^j dB(x)$. Assume that $\delta n \rightarrow C \geq 0$ as $\delta > 0$ and $n \rightarrow \infty$. Assume also that $\tilde{\rho}_2 = \lim_{n \rightarrow \infty} \rho_2(\delta)$ and $\rho_3(n)$ is a bounded sequence. Then,*

$$(2.12) \quad \mathbb{E}L_n = e^{2C/\tilde{\rho}_2} [1 + O(\delta)].$$

The proof of Theorem 2.5 uses the expansion for small δ

$$(2.13) \quad \varphi = 1 - \frac{2\delta}{\tilde{\rho}_2} + O(\delta^2).$$

obtained in Subhankulov [71], p. 326 as well as the expansion

$$(2.14) \quad 1 + \lambda \hat{B}'(\lambda - \lambda\varphi) = \delta + O(\delta^2),$$

following from the Taylor expansion of $\hat{B}'(\lambda - \lambda\varphi)$.

Using renewal theory and result (2.13), one can obtain the estimation for the stationary loss probability for the $M/GI/1/n$ queueing system with large number of servers, and the load parameter $\rho = 1 + \delta$, $\delta > 0$, as $\delta n \rightarrow C \geq 0$.

Let A_n denote the number of arrived customers during the busy cycle. According to renewal theory, for the stationary loss probability P_{loss} we have:

$$P_{loss} = \frac{\mathbb{E}L_n}{1 + \mathbb{E}A_n} = \frac{\mathbb{E}L_n}{1 + \lambda \mathbb{E}T_n}.$$

Under the assumption $\rho > 1$, from Theorem 1.1 we have the following expansion for $\mathbb{E}A_n$:

$$(2.15) \quad \mathbb{E}A_n = \frac{\rho \varphi^{-n}}{1 + \lambda \hat{B}'(\lambda - \lambda\varphi)} + \frac{\rho}{1 - \rho} + o(1).$$

Therefore under the assumption $\rho > 1$, for P_{loss} we obtain:

$$(2.16) \quad P_{loss} = \frac{(\rho - 1)^2}{\rho(\rho - 1) - \varphi^n [1 + \lambda \hat{B}'(\lambda - \lambda\varphi)]} [1 + o(\varphi^n)].$$

Under the assumption that $\rho = 1 + \delta$, $\delta > 0$, and $\delta n \rightarrow C \geq 0$ we have the following result.

Theorem 2.6. *Under the assumptions of Theorem 2.5, the following asymptotic relation holds:*

$$(2.17) \quad P_{loss} = \frac{\delta}{1 + \delta - e^{-2C/\tilde{\rho}_2}} [1 + O(\delta)].$$

Similar asymptotic relation for $M/GI/1/n$ queues, under other heavy traffic conditions where δ is a small negative parameter, has been discussed by Whitt [90].

3. CONSECUTIVE LOSSES IN $M/GI/1/n$ QUEUES

Consecutive losses are very important in performance analysis of real telecommunication systems. They have been studied in many papers, see [28], [29], [30], [35], [60], [61], [62] and many others. A convolution type recurrence relation for the distribution of k consecutive losses during a busy period (or stationary probability of k consecutive losses) has been obtained in [61] for more general systems with batch arrivals. Although the structure of these recurrence relations is simple, the notation used there looks complicated. Unfortunately, an asymptotic analysis of these recurrence relations or those ones for other numerical characteristics, as

$n \rightarrow \infty$, has not been provided neither for the systems with batch arrivals that considered in this paper [61], nor for the simpler $M/GI/1/n$ queueing systems.

We derive now the relation similar to (2.9) to the case of consecutive losses in $M/GI/1/n$ queueing systems. By k -CCL probability² we mean the (limiting) fraction of those losses belonging to sequences of k or more consecutive losses (we call them k -consecutive losses) with respect to all of the losses that occur during a long time interval. Let E_1, E_2, \dots, E_n be the states of the $M/GI/1/n$ queueing system ($n \geq 1$ and the notation for the $M/GI/1/n$ queue excludes a customer in service) at the moments of service begins, i.e. states when at the moments of service begins there are $1, 2, \dots, n$ customers in the system correspondingly. Suppose that the system is currently in the state E_j ($1 \leq j \leq n$). Then, a loss of a customer (at least one) occurs in the case when during the service time of the tagged customer, which is currently in service, there are $n - j + 2$ or more new arrivals. This probability is equal to

$$\sum_{i=n-j+2}^{\infty} \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x).$$

On the other hand, for the system that is being in state E_j , there is the probability

$$\sum_{i=n-j+k+1}^{\infty} \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x),$$

that the losses that occur during a service time, all are k -consecutive. (In this case the number of arrivals during a service time is not smaller than $n - j + k + 1$.) Let $q_j = P(E_j)$ ($q_1 + q_2 + \dots + q_n = 1$). Then, by renewal reward theorem [68] we have the following representation:

$$(3.1) \quad q_j = \frac{ET_j - ET_{j-1}}{ET_n - ET_0},$$

where ET_j , $j = 0, 1, \dots, n$, satisfy recurrence relation (2.1). By using (3.1), for the expected number of k -consecutive losses during a busy period of $M/GI/1/n$ queueing system (which is denoted by $EL_{n,k}$) we have:

$$(3.2) \quad EL_{n,k} = c_{n,k} EL_n,$$

where

$$(3.3) \quad c_{n,k} = \frac{\sum_{j=1}^n (ET_j - ET_{j-1}) \sum_{i=n-j+k+1}^{\infty} \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)}{\sum_{j=1}^n (ET_j - ET_{j-1}) \sum_{i=n-j+2}^{\infty} \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)}.$$

Note, that in the case of the $M/GI/1/0$ queueing system the result is trivial. We have:

$$EL_{0,k} = \rho \left(1 - \sum_{i=0}^{k-1} \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x) \right).$$

The value $c_{n,k}$ in (3.2) is a bounded constant. It approaches the limit as $n \rightarrow \infty$. Therefore, the asymptotic order of $EL_{n,k}$ is the same as this of EL_n . For example, in the case $\rho = 1$, $EL_{n,k} \leq 1$ is bounded for all n and therefore converges to the

²This terminology is used in [61] and [62].

limit as $n \rightarrow \infty$. (According to the explicit representation (3.3) there exists the limit of $c_{n,k}$ as $n \rightarrow \infty$.)

The *open* question concerns the local property of $EL_{n,k}$ when the values n are different and $\rho = 1$. The question is: *whether or not $EL_{n,k}$ is the same constant for all $n \geq 0$* . The complicated explicit formula (3.3) does not permit us to answer to this question easily. In all likelihood, this property does not hold in the general case. However, in the particular case of exponentially distributed service times, the answer on this question is definite: *$EL_{n,k}$ is the same for all n* . To prove this, notice first that when $\rho = 1$, we have $ET_j - ET_{j-1} = \frac{1}{\mu} = \frac{1}{\lambda}$ for all $j = 1, 2, \dots, n$. Therefore, by assuming that $B(x) = 1 - e^{-\lambda x}$ we arrive at the elementary calculations supporting this property. The exact calculations show that in this case $c_{n,k} = (\frac{1}{2})^{k-1}$. The same result can be easily obtained by an alternative way in which, under the assumption $\rho = 1$, the Markovian queueing system is represented as a symmetric random walk.

Let us now study the asymptotic behaviour of $EL_{n,k}$. The only two cases $\rho = 1$ and $\rho > 1$ are considered here. (In the case $\rho < 1$ we obtain the trivial result: $\lim_{n \rightarrow \infty} EL_{n,k} = 0$.)

Consider first the case $\rho = 1$ and $\int_0^\infty x^2 dB(x) < \infty$. In this case according to Theorem 2.3 (which is derivative from Theorem 1.3), as $j \rightarrow \infty$, we have:

$$(3.4) \quad ET_j - ET_{j-1} = \frac{2}{\lambda \rho_2} + o(1),$$

where

$$\rho_2 = \lambda^2 \int_0^\infty x^2 dB(x).$$

Therefore, in this case

$$(3.5) \quad \begin{aligned} \lim_{n \rightarrow \infty} c_{n,k} &= \lim_{n \rightarrow \infty} \frac{\sum_{j=1}^n \sum_{i=n-j+k+1}^\infty \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)}{\sum_{j=1}^n \sum_{i=n-j+2}^\infty \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)} \\ &= \frac{\sum_{i=1}^\infty i \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{i+k}}{(i+k)!} dB(x)}{\sum_{i=1}^\infty i \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{i+1}}{(i+1)!} dB(x)} \\ &= \frac{\sum_{i=1}^\infty i \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{i+k}}{(i+k)!} dB(x)}{\int_0^\infty e^{-\lambda x} dB(x)}. \end{aligned}$$

Therefore, from the known result $EL_n = 1$ for all $n \geq 0$ (Theorem 2.4) we arrive at

$$(3.6) \quad \lim_{n \rightarrow \infty} EL_{n,k} = \frac{\sum_{i=1}^\infty i \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{i+k}}{(i+k)!} dB(x)}{\widehat{B}(\lambda)}.$$

Let us now consider the case $\rho > 1$. In this case for the difference $ET_j - ET_{j-1}$, as $j \rightarrow \infty$, according to Theorem 2.1, rel. (2.2), we have the asymptotic expansion:

$$(3.7) \quad ET_j - ET_{j-1} = \frac{\rho}{\lambda[1 + \lambda \widehat{B}'(\lambda - \lambda\varphi)]} \frac{\varphi^{j-1}(1 - \varphi)}{\varphi^{2j-1}} + o(1).$$

Therefore,

$$(3.8) \quad \lim_{n \rightarrow \infty} c_{n,k} = \lim_{n \rightarrow \infty} \frac{\sum_{j=1}^n \varphi^{j-1} \sum_{i=n-j+k+1}^\infty \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)}{\sum_{j=1}^n \varphi^{j-1} \sum_{i=n-j+2}^\infty \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)}.$$

Thus, we have the following theorem.

Theorem 3.1. *If $\rho < 1$, then*

$$\lim_{n \rightarrow \infty} \text{EL}_{n,k} = 0.$$

If $\rho = 1$ and $\rho_2 = \lambda^2 \int_0^\infty x^2 dB(x) < \infty$, then

$$\lim_{n \rightarrow \infty} \text{EL}_{n,k} = \frac{\sum_{i=1}^\infty i \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{i+k}}{(i+k)!} dB(x)}{\hat{B}(\lambda)}.$$

If $\rho > 1$, then

$$\lim_{n \rightarrow \infty} \frac{\text{EL}_{n,k}}{\text{EL}_n} = \lim_{n \rightarrow \infty} \frac{\sum_{j=1}^n \varphi^{j-1} \sum_{i=n-j+k+1}^\infty \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)}{\sum_{j=1}^n \varphi^{j-1} \sum_{i=n-j+2}^\infty \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)},$$

where φ is the least positive root of equation (2.3), and EL_n is defined by asymptotic expansion (2.11) of Theorem 2.4.

4. LOSSES IN THE $GI/M/1/n$ QUEUE

Loss probabilities in $M^X/GI/1/n$ and $GI/M^Y/1/n$ queues have been studied by Miyazawa [59]. In the special case of the $GI/M/1/n$ queue, from the results of Miyazawa [59] we have as follows.

Let μ denote the parameter of service time distribution, let $A(x)$ be probability distribution function of interarrival time, and let λ be the reciprocal of the expected interarrival time. Denote the load of the system by $\rho = \frac{\lambda}{\mu}$ and let $\rho_j = \int_0^\infty (\mu x)^j dA(x)$. (Notice that $\rho_1 = \frac{1}{\rho}$.)

Miyazawa [59] proved that the loss probability $P_{loss} = P_{loss}(n)$ does exist for any ρ , and³

$$(4.1) \quad P_{loss} = \frac{1}{\sum_{j=0}^n r_j},$$

where the generating function $R(z)$ of r_j , $j = 0, 1, \dots$, is as follows:

$$(4.2) \quad R(z) = \sum_{j=0}^\infty r_j z^j = \frac{(1-z)\hat{A}(\mu - \mu z)}{\hat{A}(\mu - \mu z) - z}, \quad |z| < \varphi,$$

$\hat{A}(s) = \int_0^\infty e^{-sx} dA(x)$, and φ is the least positive root of the equation

$$(4.3) \quad z = \hat{A}(\mu - \mu z).$$

(For the least positive root of the equation we use the same notation φ as for the least positive root of similar equation (2.3). We hope, that it does not confuse the readers of this survey.)

From (4.2) we have:

$$(4.4) \quad \begin{aligned} R(z) &= \frac{(1-z)\hat{A}(\mu - \mu z)}{\hat{A}(\mu - \mu z) - z} = \frac{\hat{A}(\mu - \mu z)}{\hat{A}(\mu - \mu z) - z} - z \frac{\hat{A}(\mu - \mu z)}{\hat{A}(\mu - \mu z) - z} \\ &= \tilde{R}(z) - z\tilde{R}(z), \end{aligned}$$

³Dependence on parameter n will be omitted. It is only indicated once in the formulation of Theorem 4.3.

where

$$\tilde{R}(z) = \sum_{j=0}^{\infty} \tilde{r}_j z^j = \frac{\hat{A}(\mu - \mu z)}{\hat{A}(\mu - \mu z) - z}.$$

Note that

$$\begin{aligned} r_0 &= \tilde{r}_0, \\ r_{j+1} &= \tilde{r}_{j+1} - \tilde{r}_j, \quad j \geq 0. \end{aligned}$$

Therefore,

$$\sum_{j=0}^n r_j = \tilde{r}_n$$

and therefore from (4.1) we have

$$(4.5) \quad P_{loss} = \frac{1}{\tilde{r}_n}.$$

The sequence \tilde{r}_n satisfies the convolution type recurrence relation of (1.2). Specifically,

$$(4.6) \quad \tilde{r}_n = \sum_{j=0}^n \tilde{r}_{n-j+1} \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^j}{j!} dA(x).$$

Therefore by applying Theorem 1.1 we arrive at the following theorem (for details of the proof see [9]).

Theorem 4.1. *If $\rho < 1$, then as $n \rightarrow \infty$*

$$(4.7) \quad P_{loss} = \frac{(1 - \rho)[1 + \mu \hat{A}'(\mu - \mu \varphi)] \varphi^n}{1 - \rho - \rho[1 + \mu \hat{A}'(\mu - \mu \varphi)] \varphi^n} + o(\varphi^{2n}).$$

If $\rho = 1$ and $\rho_2 < \infty$, then

$$(4.8) \quad \lim_{n \rightarrow \infty} n P_{loss} = \frac{\rho_2}{2}.$$

If $\rho > 1$, then

$$(4.9) \quad \lim_{n \rightarrow \infty} P_{loss} = \frac{\rho - 1}{\rho}.$$

Using Tauberian Theorems 1.2 and 1.3 one can improve (4.8). In this case we have the following two theorems.

Theorem 4.2. *Assume that $\rho = 1$ and $\rho_2 < \infty$. Then, as $n \rightarrow \infty$,*

$$(4.10) \quad P_{loss} = \frac{\rho_2}{2n} + O\left(\frac{\log n}{n^2}\right).$$

The proof of this theorem follows immediately from Theorem 1.2.

Theorem 4.3. *Assume that $\rho = 1$ and $\rho_2 < \infty$. Then, as $n \rightarrow \infty$,*

$$(4.11) \quad \frac{1}{P_{loss}(n+1)} - \frac{1}{P_{loss}(n)} = \frac{2}{\rho_2} + o(1).$$

Proof. The proof of this theorem is similar to that of the proof of Theorem 2.3. \square

Choi and Kim [25] and Choi, Kim and Wee [26] study the asymptotic behavior of the stationary probabilities and loss probabilities. Some of these results are close to the results obtained in Theorems 4.1 and 4.3. For more details see the discussion section of [9].

The further heavy traffic analysis leads to the following two theorems.

Theorem 4.4. *Let $\rho = 1 - \delta$, $\delta > 0$, and let $\delta n \rightarrow C > 0$ as $n \rightarrow \infty$ and $\delta \rightarrow 0$. Assume that $\rho_3(n)$ is a bounded sequence, and there exists $\tilde{\rho}_2 = \lim_{n \rightarrow \infty} \rho_2(n)$. Then,*

$$(4.12) \quad P_{loss} = \frac{\delta e^{-2C/\tilde{\rho}_2}}{1 - e^{-2C/\tilde{\rho}_2}} [1 + o(1)].$$

Theorem 4.5. *Let $\rho = 1 - \delta$, $\delta > 0$, and let $\delta n \rightarrow 0$ as $n \rightarrow \infty$ and $\delta \rightarrow 0$. Assume that $\rho_3(n)$ is a bounded sequence, and there exists $\tilde{\rho}_2 = \lim_{n \rightarrow \infty} \rho_2(n)$. Then,*

$$(4.13) \quad P_{loss} = \frac{\tilde{\rho}_2}{2n} + o\left(\frac{1}{n}\right).$$

The proof of the both of these theorems is based on asymptotic expansions, which are analogous to those of (2.13) and (2.14).

Similar results, related to the heavy traffic analysis of $GI/M/1/n$ queues, have been obtained in [90].

5. LOSSES IN THE $GI/M/m/n$ QUEUE

Takács' theorem has also been applied for analysis of the loss probabilities in multiserver $GI/M/m/n$ queueing systems [11]. An application of Takács' theorem in this case, however, is not entirely straightforward and based on special approximations. Specifically, the recurrence relation of convolution type (1.2) is valid only in limit, and a technically hard analytic proof with complicated notation is required in order to reduce the equations describing the stationary loss probability to the asymptotic recurrence relation of convolution type (1.2).

In this section we do not present the details of the proofs. These details of the proofs can be found in [11]. We only formulate the theorems and discuss general features and differences between the $GI/M/m/n$ and $GI/M/1/n$ cases in the corresponding theorems.

As in the case $m = 1$ above, $A(x)$ is the probability distribution function of interarrival time, λ is the reciprocal of the expected interarrival time, $\hat{A}(s)$ is the Laplace-Stieltjes transform of $A(x)$, where the argument s is assumed to be nonnegative. The parameter of the service time distribution is denoted by μ , and the load of the system $\rho = \lambda/(m\mu)$. The least positive root of the equation $z = \hat{A}(m\mu - m\mu z)$ will be denoted φ_m . The loss probability P_{loss} is now dependent of m and n .

The following theorem on the stationary loss probability have been established in Abramov [11].

Theorem 5.1. *If $\rho > 1$, then for any $m \geq 1$,*

$$(5.1) \quad \lim_{n \rightarrow \infty} P_{loss} = \frac{\rho - 1}{\rho}.$$

If $\rho = 1$ and $\rho_2 = \int_0^\infty (\mu x)^2 dA(x) < \infty$, then for any $m \geq 1$,

$$(5.2) \quad \lim_{n \rightarrow \infty} n P_{loss} = \frac{\rho_2}{2}.$$

If $\rho = 1$ and $\rho_3 = \int_0^\infty (mx)^3 dA(x) < \infty$, then for large n and any $m \geq 1$,

$$(5.3) \quad P_{loss} = \frac{\rho_2}{2n} + O\left(\frac{\log n}{n^2}\right).$$

If $\rho < 1$, then for P_{loss} we have the limiting relation:⁴

$$(5.4) \quad P_{loss} = K_m \frac{(1-\rho)[1 + m\mu\hat{A}'(m\mu - m\mu\varphi_m)]\varphi_m^{n-1}}{1 - \rho - \rho[1 + m\mu\hat{A}'(m\mu - m\mu\varphi_m)]\varphi_m^{n-1}} + o(\varphi_m^{2n}),$$

where

$$(5.5) \quad K_m = \left[1 + (1 - \varphi_m) \sum_{j=1}^m \binom{m}{j} \frac{C_j}{1 - \sigma_j} \cdot \frac{m(1 - \sigma_j) - j}{m(1 - \varphi_m) - j} \right]^{-1},$$

$$\sigma_j = \int_0^\infty e^{-j\mu x} dA(x),$$

$$C_j = \prod_{i=1}^j \frac{1 - \sigma_i}{\sigma_i}.$$

Relations (5.1), (5.2) and (5.4) follow from Takács' theorem 1.1, relation (5.3) follows from Tauberian theorem 1.2. Relations (5.1), (5.2) and (5.3) are as the corresponding results for the $GI/M/1/n$ queue. However, there is the difference between (5.4) and (4.7) associated with special representation (5.5) of the coefficient K_m , so the analysis of the case $\rho < 1$ is more delicate than that in the cases $\rho > 1$ and $\rho = 1$.

By alternative methods, the asymptotic loss and stationary probabilities in $GI/M/m/n$ queues have been studied in several papers. The analytic proofs given in these papers are much more difficult than those by application of Takács' theorem 1.1. We refer Kim and Choi [48], Choi et al. [27] and Simonot [70], where the readers can find the proofs by using the standard analytic techniques.

It is surprising that in the heavy traffic case, where ρ approaches 1 from the below, we obtain the same asymptotic representation for the loss probability as in the $GI/M/1/n$ case. Although the expression for K_m (5.5) looks complicated, its asymptotic expansion when $\rho = 1 - \delta$, $\delta > 0$, and $\delta n \rightarrow C$ as $n \rightarrow \infty$ and $\delta \rightarrow 0$ is very simple:

$$K_m = 1 + O(\delta).$$

Therefore, we arrive at the following theorem.

Theorem 5.2. *Let $\rho = 1 - \delta$, $\delta > 0$, and let $\delta n \rightarrow C$ as $n \rightarrow \infty$ and $\delta \rightarrow 0$. Suppose that $\rho_3 = \rho_3(n)$ is a bounded sequence, and there exists $\tilde{\rho}_2 = \lim_{n \rightarrow \infty} \rho_2(n)$. Then, in the case $C > 0$ for any $m \geq 1$ we have*

$$(5.6) \quad P_{loss} = \frac{\delta e^{-2C/\tilde{\rho}_2}}{1 - e^{-2C/\tilde{\rho}_2}} [1 + o(1)].$$

In the case $C = 0$ for any $m \geq 1$ we have

$$(5.7) \quad P_{loss} = \frac{\tilde{\rho}_2}{2n} + o\left(\frac{1}{n}\right).$$

⁴The asymptotic relation, which is presented here, is more exact than that was presented in [11] in the formulation of Theorem 3.1 of that paper.

It is readily seen that the asymptotic representation given by (5.6) and (5.7) are exactly the same as the corresponding asymptotic representations of (4.12) and (4.13).

6. FUTURE RESEARCH PROBLEMS

In the previous sections, applications of Takács' theorem to queueing systems have been discussed. In this section, which concludes Part 1, we formulate problems for the future applications of Takács' theorem to different queueing systems. Specifically, we discuss possible ways of application of Takács theorem to retrial queueing systems.

Consider a single server queueing system with Poisson input rate λ and n number of waiting places. A customer, who upon arrival finds all waiting places busy, goes to the secondary system, and after some random time arrives at the main system again. If at least one of waiting places is free, the customer joins the main queue. Assuming that service times in the main system are independent and identically distributed random variables, one can interest in asymptotic behavior of the fraction of retrials with respect to the total number of arrivals during a busy period.

Assumptions on retrials can be very different. One of the simplest models is based on the assumption that times between retrials are independent identically distributed random variables, and their distribution is independent of the number of customer in the secondary system. In another model times between retrials are exponentially distributed with parameter depending on the number of customers in the secondary queue. The most familiar model amongst them is a model with linear retrial policy: If the number of customers in the secondary system is j , then the retrial rate is $j\mu$.

Asymptotic analysis as $n \rightarrow \infty$ of the fraction of retrials with respect to the total number of customers arrived during a busy period is a significant performance characteristic of these large retrial queueing systems. An application of Takács' theorem should be based on comparison of the desired characteristics of retrial queueing systems with the corresponding characteristics of the $M/GI/1/n$ queueing system with losses. Specifically, denoting by L_n the number of losses in the $M/GI/1/n$ queueing system and by R_n the number of retrials in a given queueing system with retrials, we interest in finding the values c_n supporting the equality $ER_n = c_n EL_n$. If the properties of the sequence c_n are well-specified (for example, there exists a limit or an appropriate estimate), then the application of Takács' theorem to ER_n as $n \rightarrow \infty$ becomes elementary. (The asymptotic behavior of EL_n is given by Theorem 2.4.)

Part 2. Applications to stochastic models of communication systems and dam/storage systems

7. ASYMPTOTIC ANALYSIS OF THE NUMBER OF LOST MESSAGES IN COMMUNICATION SYSTEMS

In this section we study losses in optical telecommunication networks, where we develop the results on losses in $M/GI/1/n$ queues considered in Section 1. We discuss the results established in [10].

Long messages being sent are divided into a random number of packets which are transmitted independently of one another. An error in transmission of a packet

results in a loss of the entire message. Messages arrive to the $M/GI/1$ finite buffer model (the exact description of the model is given below) and can be lost in two cases as either at least one of its packets is corrupted or the buffer is overflowed.

The model is the following extension of the usual $M/GI/1/n$ system. We consider queueing system with Poisson input rate λ of batch arrivals. The system serves each of these batches, and each service time has probability distribution $B(x)$ with mean $\frac{1}{\mu}$. The random batches κ_i are bounded from the above and below, so that

$$(7.1) \quad P\{\kappa^{lower} \leq \kappa_i \leq \kappa^{upper}\} = 1.$$

κ_i is the number of packets associated with the i th message.

Let us denote

$$\zeta = \sup \left\{ m : \sum_{i=1}^m \kappa_i \leq N \right\},$$

and according to assumption (7.1) there are two fixed values ζ^{lower} and ζ^{upper} depending on N and $P\{\zeta^{lower} \leq \zeta \leq \zeta^{upper}\} = 1$. Let ξ_i be the number of messages in the queue immediately before arrival of the i th message. Then the message is lost if $\xi_i > \zeta_i$. Otherwise it joins the queue. ζ_i is the i th (generic) random level in terms of a number of possible messages in the system. The special case when $P\{\kappa_i = l\} = 1$ (a message contains a fixed (non-random) number of packets) leads to the standard $M/GI/1/n$ queueing system, where $n = \lfloor \frac{N}{l} \rfloor$ is the integer part of $\frac{N}{l}$.

It is also assumed that each message is marked with probability p , and we study the asymptotic behavior of the loss probability under assumptions that $E\zeta$ increases to infinity and p vanishes. The lost probability is the probability that the message is either marked or lost because of overflowing the queue. We demonstrate an application of Takács' theorem as well as Theorems 1.2 and 1.3 for the solution of all of these problems.

The queueing system described above is not standard, and the explicit representation for its characteristics can not be obtained traditionally. We will introduce a class Σ of queueing systems. A simple representative of this class is the system S_1 which has been determined above for which we will establish the balance equation for the expectations of accepted/rejected customers in the system which are similar to those of (2.6) and (2.7) in the case of the standard $M/GI/1/n$ system. Such equations will be written explicitly in the sequel.

In order to define the class Σ let us first study elementary processes of the system S_1 .

Let ξ_i denote the number of messages in the system S_1 immediately before arrival of the i th message, $\xi_1 = 0$, and let s_i denote the number of service completions between the i th and $i + 1$ st arrivals. Clearly, that

$$(7.2) \quad \xi_{i+1} = \xi_i - s_i + 1_{\{\xi_i \leq \zeta_i\}},$$

where the term $1_{\{\xi_i \leq \zeta_i\}}$ in (7.2) indicates that the i th message is accepted. Obviously, that s_i is not greater than $\xi_i + 1_{\{\xi_i \leq \zeta_i\}}$.

Let us consider now a new queueing system as above with the same rate of Poisson input λ and the same probability distribution function of the service time $B(x)$, but with another sequence $\tilde{\zeta}_1, \tilde{\zeta}_2, \dots$ of arbitrary dependent sequence of random variables all having the same distribution as ζ . Let $\tilde{\xi}_i$ denote the number of messages immediately before arrival of the i th message ($\tilde{\xi}_1 = 0$), and let \tilde{s}_i denote

the number of service completions between the i th and $i + 1$ st arrival. Analogously to (7.2) we have

$$(7.3) \quad \tilde{\xi}_{i+1} = \tilde{\xi}_i - \tilde{s}_i + 1_{\{\tilde{\xi}_i \leq \tilde{\zeta}_i\}}.$$

Definition 7.1. The queueing system \mathcal{S} is said to belong to the set Σ of queueing systems if $E\tilde{\xi}_i = E\xi_i$, $E\tilde{s}_i = Es_i$ and

$$P\{\tilde{\xi}_i \leq \tilde{\zeta}_i\} = P\{\xi_i \leq \zeta_i\} \text{ for all } i \geq 1.$$

Let us now consider an example of queueing system belonging to the set Σ . The example is $\tilde{\zeta}_1 = \tilde{\zeta}_2 = \dots$. Denote this queueing system by \mathcal{S}_2 . This example is artificial, but it helps to easily study this specific system, and together with this system all of the systems belonging to this class Σ .

Specifically, for this system \mathcal{S}_2 according to the induction for all $i \geq 1$ we have:

$$(7.4) \quad E\tilde{s}_i = Es_i,$$

$$(7.5) \quad P\{\tilde{\xi}_i \leq \tilde{\zeta}_i\} = P\{\xi_i \leq \zeta_i\},$$

and

$$(7.6) \quad E\tilde{\xi}_i - E\tilde{s}_i + P\{\tilde{\xi}_i \leq \tilde{\zeta}_i\} = E\xi_i - Es_i + P\{\xi_i \leq \zeta_i\}.$$

Relations (7.4), (7.5) and (7.6) enable us to conclude that $\mathcal{S}_2 \in \Sigma$. Furthermore, from these relations (7.4), (7.5) and (7.6) all of characteristics of all of queueing systems from the class Σ do exist and are the same. Therefore, for our conclusion it is enough to study the queueing system \mathcal{S}_2 , which is the simplest than all other.

Let \tilde{T}_ζ denote a busy period of system \mathcal{S}_2 . By the formula for the total expectation

$$(7.7) \quad E\tilde{T}_\zeta = \sum_{i=\zeta^{lower}}^{\zeta^{upper}} ET_i P\{\zeta = i\},$$

where ET_i is the expectation of the busy period of an $M/GI/1/i$ queueing system with the same interarrival and service time distributions. The expectations ET_i are determined from the convolution type recurrence relations of (2.1), and all of the results of the above theory related to $M/GI/1/n$ can be applied here. Then one can write $ET_\zeta = E\tilde{T}_\zeta$, and therefore for the queueing system \mathcal{S}_1 we have the same relation as (7.7):

$$(7.8) \quad ET_\zeta = \sum_{i=\zeta^{lower}}^{\zeta^{upper}} ET_i P\{\zeta = i\},$$

Along with the notation T_ζ for the busy period of the queueing system \mathcal{S}_1 we consider also the following characteristics of this queueing system. Let I_ζ denote an idle period, and let P_ζ , M_ζ and R_ζ denote the number of processed messages, the number of marked messages and the number of refused messages respectively. We will use the following terminology. The term *refused* message is used for the case of overflowing the buffer, while the term *lost* message is used for the case where a message is either refused or marked. The number of lost messages during a busy period is denoted by L_ζ . Analogously, by lost probability we mean the probability when the arrival message is lost.

Lemma 7.2. *For the expectations ET_ζ , EP_ζ , EM_ζ , ER_ζ we have the following representations:*

$$(7.9) \quad EP_\zeta = \mu ET_\zeta,$$

$$(7.10) \quad EM_\zeta = pEP_\zeta,$$

$$(7.11) \quad ER_\zeta = (\rho - 1)EP_\zeta + 1.$$

The proof of this lemma is based on Wald's equations [41], p.384 and similar to the proof of relation (2.8) from the system of equations (2.6) and (2.7). The relations (7.8), (7.9), (7.10) and (7.11) all together define all of the required expectations, and all of them are expressed via recurrence relation of convolution type. Therefore, one can apply Takács' theorem and Tauberian theorems 1.2 and 1.3.

Denote: $\rho_j = \int_0^\infty (\lambda x)^j dB(x)$, $j = 1, 2, \dots$, where according to the earlier notation $\rho = \rho_1$ is the load parameter of the system.

We write $\zeta = \zeta(N)$ to point out the dependence on parameter N . As N tends to infinity, both ζ^{lower} and ζ^{upper} tend to infinity, and together with them $\zeta(N)$ tends to infinity almost surely (a.s.) For the above characteristics of the queueing system we have the following theorems.

Theorem 7.3. *If $\rho < 1$, then*

$$(7.12) \quad \lim_{N \rightarrow \infty} EP_{\zeta(N)} = \frac{1}{1 - \rho}.$$

If $\rho = 1$ and $\rho_2 < \infty$, then

$$(7.13) \quad \lim_{N \rightarrow \infty} \frac{EP_{\zeta(N)}}{E\zeta(N)} = \frac{2}{\rho_2}.$$

If $\rho > 1$, then

$$(7.14) \quad \lim_{N \rightarrow \infty} \left[EP_{\zeta(N)} - \frac{1}{E\varphi^{\zeta(N)}[1 + \lambda \widehat{B}'(\lambda - \lambda\varphi)]} \right] = \frac{1}{1 - \rho},$$

where $\widehat{B}(s)$ is the Laplace-Stieltjes transform of the probability distribution function $B(x)$ and φ is the least positive root of equation $z = \widehat{B}(\lambda - \lambda z)$. (See rel. (2.3).)

Theorem 7.4. *If $\rho = 1$ and $\rho_3 < \infty$, then*

$$(7.15) \quad EP_{\zeta(N)} = \frac{2}{\rho_2} + O(\log N).$$

Theorem 7.5. *If $\rho < 1$, then*

$$(7.16) \quad \lim_{N \rightarrow \infty} ER_{\zeta(N)} = 0.$$

If $\rho = 1$, then for all $N \geq 0$

$$(7.17) \quad ER_{\zeta(N)} = 1.$$

If $\rho > 1$, then

$$(7.18) \quad \lim_{N \rightarrow \infty} \left[ER_{\zeta(N)} - \frac{\rho - 1}{E\varphi^{\zeta(N)}[1 + \lambda \widehat{B}'(\lambda - \lambda\varphi)]} \right] = 0.$$

Theorems 7.3 and 7.4 follow from the corresponding Takács' theorem and Postnikov's Tauberian theorem 1.2. Theorem 7.5 follows from Takács' theorem and is an analogue of Theorem 2.4 on losses in $M/GI/1/n$ queues. Relation (7.17) says that the remarkable property of losses under the condition $\rho = 1$ remains the same as in the case of the standard $M/GI/1/n$ queueing system. The details of proofs for all of these theorems as well as the following theorems of this section can be found in [10].

In the case where the number of packets in each message is considered to be fixed, then from Postnikov's Tauberian Theorem 1.3 we have as follows.

Theorem 7.6. *If $\rho = 1$ and $\rho_2 < \infty$, then as $n \rightarrow \infty$,*

$$(7.19) \quad EP_{n+1} - EP_n = \frac{2}{\rho_2} + o(1),$$

where the index $n + 1$ says that P_{n+1} is the number of processed messages during a busy period of the $M/GI/1/n + 1$ queueing system.

Note, that the proof of Theorem 7.6 is based on application of Tauberian Theorem 1.3 and is exactly the same as the proof of Theorem 4.3.

Under heavy traffic conditions we have as follows.

Theorem 7.7. *Let $\rho = 1 + \delta$, $\delta > 0$, and $\delta\zeta(N) \rightarrow C > 0$ a.s. as $N \rightarrow \infty$ and $\delta \rightarrow 0$. Assume also that $\rho_3 = \rho_3(N)$ is a bounded sequence, and there exists $\tilde{\rho}_2 = \lim_{N \rightarrow \infty} \rho_2(N)$. Then,*

$$(7.20) \quad EP_{\zeta(N)} = \frac{e^{2C/\tilde{\rho}_2} - 1}{\delta} + O(1),$$

$$(7.21) \quad ER_{\zeta(N)} = e^{2C/\tilde{\rho}_2} + o(1).$$

Theorem 7.8. *Let $\rho = 1 + \delta$, $\delta > 0$ and $\delta\zeta(N) \rightarrow 0$ a.s. as $N \rightarrow \infty$ and $\delta \rightarrow 0$. Assume also that $\rho_3 = \rho_3(N)$ is a bounded sequence, and there exists $\tilde{\rho}_2 = \lim_{N \rightarrow \infty} \rho_2(N)$. Then,*

$$(7.22) \quad EP_{\zeta(N)} = \frac{2}{\tilde{\rho}_2} E\zeta(N) + O(1),$$

$$(7.23) \quad ER_{\zeta(N)} = 1 + o(1).$$

Asymptotic behavior of the loss probability can be deduced from the above asymptotic theorems by using renewal arguments. According to renewal arguments, the loss probability is

$$(7.24) \quad \Pi_\zeta = \frac{EL_\zeta}{ER_\zeta + EP_\zeta} = \frac{ER_\zeta + EM_\zeta}{ER_\zeta + EP_\zeta} = \frac{ER_\zeta + pEP_\zeta}{ER_\zeta + EP_\zeta},$$

where p is the probability of losing a message, because one of its packets is corrupted.

We have as follows.

Theorem 7.9. *If $\rho < 1$, then*

$$(7.25) \quad \lim_{N \rightarrow \infty} \Pi_{\zeta(N)} = p.$$

Limiting relation (7.25) is also valid when $\rho = 1$ and $\rho_2 < \infty$. If $\rho > 1$, then

$$(7.26) \quad \Pi_{\zeta(N)} = \frac{p + \rho - 1}{\rho} \cdot \frac{(\rho - 1) + p[1 + \lambda\widehat{B}'(\lambda - \lambda\varphi)]E\varphi^{\zeta(N)}}{(\rho - 1) + [1 + \lambda\widehat{B}'(\lambda - \lambda\varphi)]E\varphi^{\zeta(N)}} + o(E\varphi^{\zeta(N)}).$$

Theorem 7.10. *If $\rho = 1$ and $\rho_3 < \infty$, then as $N \rightarrow \infty$,*

$$(7.27) \quad \Pi_{\zeta(N)} = p + \frac{(1-p)\rho_2}{2E\zeta(N)} + O\left(\frac{\log N}{N^2}\right).$$

If under the assumptions of Theorem 7.9 to assume additionally that $p \rightarrow 0$, then in the case $pN \rightarrow C > 0$ as $p \rightarrow 0$ and $N \rightarrow \infty$ we have:

$$\Pi_{\zeta(N)} = \frac{C}{N} + \frac{\rho_2}{2E\zeta(N)} + O\left(\frac{\log N}{N^2}\right).$$

In the case $pN \rightarrow 0$ as $p \rightarrow 0$ and $N \rightarrow \infty$ we have:

$$\Pi_{\zeta(N)} = \frac{\rho_2}{2E\zeta(N)} + O\left(p + \frac{\log N}{N^2}\right).$$

Theorem 7.11. *Let $\rho = 1 + \delta$, $\delta > 0$, and $\delta\zeta(N) \rightarrow C > 0$ as $N \rightarrow \infty$ and $\delta > 0$, and let $p \rightarrow 0$. Assume also that $\rho_3 = \rho_3(N)$ is a bounded sequence, and there exists $\tilde{\rho}_2 = \lim_{N \rightarrow \infty} \rho_2(N)$.*

(i) *If $\frac{p}{\delta} \rightarrow D \geq 0$, then*

$$(7.28) \quad \Pi_{\zeta(N)} = \left(D + \frac{e^{2C/\tilde{\rho}_2}}{e^{2C/\tilde{\rho}_2} - 1}\right) \delta + o(\delta).$$

(ii) *If $\frac{p}{\delta} \rightarrow \infty$, then*

$$(7.29) \quad \Pi_{\zeta(N)} = p + O(\delta).$$

Theorem 7.12. *Let $\rho = 1 + \delta$, $\delta > 0$, and $\delta\zeta(N) \rightarrow 0$ as $N \rightarrow \infty$ and $\delta > 0$, and let $p \rightarrow 0$. Assume also that $\rho_3 = \rho_3(N)$ is a bounded sequence, and there exists $\tilde{\rho}_2 = \lim_{N \rightarrow \infty} \rho_2(N)$.*

(i) *If $\frac{p}{\delta} \rightarrow D \geq 0$, then*

$$(7.30) \quad \Pi_{\zeta(N)} = p + \frac{\tilde{\rho}_2}{2E\zeta(N)} + o\left(\frac{1}{N}\right).$$

(ii) *If $\frac{p}{\delta} \rightarrow \infty$, then we have (7.29).*

In the special case where each message contains exactly l packets, $n = \lfloor \frac{N}{l} \rfloor$, we have as follows.

Theorem 7.13. *If $\rho_1 = 1$ and $\rho_2 < \infty$, then as $n \rightarrow \infty$*

$$\Pi_{n+1} - \Pi_n = \frac{\frac{1}{n(n+1)} \frac{2}{\rho_2} (p-1)}{\left(\frac{2}{\rho_2} + \frac{1}{n+1}\right) \left(\frac{2}{\rho_2} + \frac{1}{n}\right)} + o\left(\frac{1}{n^2}\right).$$

Asymptotic Theorems 7.9 – 7.12 on loss probability enable us to make conclusion on adding redundant packets into messages. The standard assumption given in [10] is that a redundant packet decreases probability p by γ times but increases the load of the system by $\tilde{\gamma}$ times. The analysis in [10] showed that in the case $\rho < 1$ the adding a number of redundant packets can decrease the loss probability with geometric rate while $\rho \leq 1$. It was also shown that in critical cases where $\rho = 1 + \delta$ adding redundancy can be profitable as well. The details of the analysis can be found in [10].

There is a large number of papers in the literature on communication systems and networks studying related problems on analysis of losses and adding redundancy. We refer [16], [17], [31], [36], [37], [42] to only mention a few. All of these papers use analytic techniques for the solution of one or other problem.

8. NEW ASYMPTOTIC RESULTS FOR THE NUMBER OF LOST MESSAGES IN TELECOMMUNICATION SYSTEMS

In this section we continue to study the asymptotic behavior of the number of lost messages in optical telecommunication systems. More specifically, in this section we apply the results of Sect. 3 to establish the asymptotic results for consecutive refused messages of the model described in the previous section. As in the previous section the phrase *refused message* is used to indicate that the message is lost by overflowing the buffer, and in this section we provide new results just for *consecutive refused messages*. For the number of k -consecutive refused messages we use the notation $R_{\zeta(N),k}$, i.e. we only add the index k to the previous notation $R_{\zeta(N)}$ used in the previous section.

According to Lemma 7.2

$$ER_{\zeta(N)} = (\rho - 1)EP_{\zeta(N)},$$

and

$$EP_{\zeta(N)} = \mu ET_{\zeta(N)}$$

(see rel. (7.9) and (7.11)), where $P_{\zeta(N)}$ is the number of processed messages during a busy period $T_{\zeta(N)}$.

The expectation $ET_{\zeta(N)}$ is in turn found by using the total probability formula

$$ET_{\zeta(N)} = \sum_{i=\zeta^{lower}}^{\zeta^{upper}} ET_i P\{\zeta(N) = i\}$$

(see rel. (7.8)), where T_i is the length of a busy period in the $M/GI/1/i$ queueing system. The asymptotic analysis of $ET_{\zeta(N)}$ and $ER_{\zeta(N)}$ is based on asymptotic behavior of ET_n and the results of that analysis for $ER_{\zeta(N)}$ is given by Theorem 7.5. Similarly to (7.8), the above total probability formula can be written for $ER_{\zeta(N)}$, so we have:

$$ER_{\zeta(N)} = \sum_{i=\zeta^{lower}}^{\zeta^{upper}} EL_i P\{\zeta(N) = i\},$$

where L_i denotes the number of losses during a busy period of the $M/GI/1/i$ queueing system. Therefore, the results of Sect. 3 can be applied immediately here.

Theorem 8.1. *If $\rho < 1$, then*

$$\lim_{N \rightarrow \infty} ER_{\zeta(N),k} = 0.$$

If $\rho = 1$ and $\rho_2 = \lambda^2 \int_0^\infty x^2 dB(x) < \infty$, then

$$\lim_{N \rightarrow \infty} ER_{\zeta(N),k} = \frac{\sum_{i=1}^{\infty} i \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{i+k}}{(i+k)!} dB(x)}{\widehat{B}(\lambda)}.$$

If $\rho > 1$, then

$$\lim_{N \rightarrow \infty} \frac{ER_{\zeta(N),k}}{ER_{\zeta(N)}} = \lim_{N \rightarrow \infty} \frac{\sum_{j=1}^N \varphi^{j-1} \sum_{i=N-j+k+1}^{\infty} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)}{\sum_{j=1}^N \varphi^{j-1} \sum_{i=N-j+2}^{\infty} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB(x)},$$

where φ is the least positive root of equation (2.3), and $ER_{\zeta(N)}$ is defined by asymptotic expansion (7.18) of Theorem 7.5.

Other new results on asymptotic behavior of $ER_{\zeta(N),k}$ as $N \rightarrow \infty$ can be obtained by the further analysis of the results obtained in Sect. 3 and 7. Similarly to Theorems 7.7 and 7.8 we have the following theorem.

Theorem 8.2. *Let $\rho = 1 + \delta$, $\delta > 0$, and $\delta\zeta(N) \rightarrow C \geq 0$ a.s. as $N \rightarrow \infty$ and $\delta \rightarrow 0$. Assume also that $\rho_3 = \rho_3(N)$ is a bounded sequence, and there exists $\tilde{\rho}_2 = \lim_{N \rightarrow \infty} \rho_2(N)$. Then,*

$$ER_{\zeta(N),k} = e^{2C/\tilde{\rho}_2} \frac{\sum_{i=1}^{\infty} i \int_0^{\infty} e^{-\lambda x} \frac{(\lambda x)^{i+k}}{(i+k)!} dB(x)}{\hat{B}(\lambda)} + o(1).$$

9. OPTIMAL POLICIES OF USING WATER IN LARGE DAMS: A SIMPLIFIED MODEL

In this section we discuss an application of Takács' theorem to optimal control of a large dam. We consider a simple variant of this problem given in [12]. The extended version corresponding to [13] will be discussed in the next section.

The upper and lower levels of a dam are denoted L^{upper} and L^{lower} correspondingly, and the difference between the upper and lower level $n = L^{upper} - L^{lower}$ characterizes the capacity of this dam. The value n is assumed to be large, and this assumption enables us to use asymptotic analysis as $n \rightarrow \infty$.

We assume that the units of water arriving at the dam are registered at random instant t_1, t_2, \dots , and the interarrival times, $\tau_n = t_{n+1} - t_n$, are mutually independent, exponentially distributed random variables with parameter λ . Outflow of water is state dependent as follows. If the level of water is between L^{lower} and L^{upper} then the dam is said to be in normal state and the duration time between departure of a unit of water has the probability distribution function $B_1(x)$. If the level of water increases the level L^{upper} , then the probability distribution function of the interval between unit departures is $B_2(x)$. If the level of water is exactly L^{lower} , then the departure process of water is frozen, and it is resumed again as soon as the level of water exceeds the value L^{lower} .

In terms of queueing theory, the problem can be formulated as follows. Consider a single-server queueing system with Poisson arrival stream of rate λ . The service time of customers depends upon queue-length as follows. If at the moment of a beginning of a service, the number of customers in the system is not greater than n , then this customer is served by probability distribution function $B_1(x)$. Otherwise (if at the moment of a beginning of a service, the number of customers in the system exceeds n), then the probability distribution function of the service time of this customer is $B_2(x)$. Notice, that the lower level L^{lower} is associated with an empty dam. The dam specification of the problem is characterized by performance criteria, which in queueing formulation is as follows.

Let $Q(t)$ denote the (stationary) queue-length in time t . The problem is to choose the output parameter of the system (parameter of service time distribution $B_1(x)$) so to minimize the functional

$$(9.1) \quad J(n) = p_1(n)J_1(n) + p_2(n)J_2(n),$$

where $p_1(n) = \lim_{t \rightarrow \infty} P\{Q(t) = 0\}$, $p_2(n) = \lim_{t \rightarrow \infty} P\{Q(t) > n\}$, and $J_1(n)$, $J_2(n)$ are the corresponding damage costs proportional to n . To be correct, assume that $J_1(n) = j_1 n$ and $J_2(n) = j_2 n$, where j_1 and j_2 are given positive values.

Assuming that $n \rightarrow \infty$ we will often write the p_1 and p_2 without index n . The index n will be also omitted in other functions such as J_1 , J_2 and so on.

We will assume that the input rate λ and probability distribution function $B_2(x)$ are given. The unknown parameter in the control problem is associated with probability distribution function $B_1(x)$. We will assume that $B_1(x) = B_1(x, C)$ is the family of probability distributions depending on parameter $C \geq 0$, and this parameter C is closely related to the expectation $\int_0^\infty x dB_1(x)$. Then the output rate of the system in the normal state is associated with the choice of parameter $C \geq 0$ resulting the choice of $B_1(x, C)$.

We use the notation $\frac{1}{\mu_i} = \int_0^\infty x dB_i(x)$, and $\rho_i = \frac{\lambda}{\mu_i}$ for $i = 1, 2$. We assume that $\rho_2 < 1$, and this last assumption provides the stationary behavior of queueing system and existence of required limiting stationary probabilities p_1 and p_2 (independent of an initial state of the process). In addition, we assume existence of the third moment, $\rho_{1,k} = \int_0^\infty (\lambda x)^k dB_1(x) < \infty$, $k = 2, 3$.

For the above state dependent queueing system introduce the notation as follows. Let T_n , I_n , and ν_n respectively denote the duration of a busy period, the duration of an idle period, and the number of customers served during a busy period. Let $T_n^{(1)}$ and $T_n^{(2)}$ respectively denote the total times during a busy period when $0 < Q(t) \leq n$ and $Q(t) > n$, and let $\nu_n^{(1)}$ and $\nu_n^{(2)}$ respectively denote the total numbers of customers served during a busy period when $0 < Q(t) \leq n$ and $Q(t) > n$.

We have the following two equations

$$(9.2) \quad ET_n = ET_n^{(1)} + ET_n^{(2)},$$

$$(9.3) \quad E\nu_n = E\nu_n^{(1)} + E\nu_n^{(2)}.$$

From Wald's equation [41], p.384, we obtain:

$$(9.4) \quad ET_n^{(1)} = \frac{1}{\mu_1} E\nu_n^{(1)},$$

$$(9.5) \quad ET_n^{(2)} = \frac{1}{\mu_2} E\nu_n^{(2)}.$$

The number of arrivals during a busy cycle coincides with the total number of served customers during a busy period. By using Wald's equation again we have

$$(9.6) \quad \begin{aligned} E\nu_n^{(1)} + E\nu_n^{(2)} &= \lambda ET_n + \lambda EI_n \\ &= \lambda ET_n + 1 \\ &= \lambda (ET_n^{(1)} + ET_n^{(2)}) + 1. \end{aligned}$$

Substituting (9.4) and (9.5) for the right-hand side of (9.6), we obtain:

$$(9.7) \quad E\nu_n^{(1)} + E\nu_n^{(2)} = \rho_1 E\nu_n^{(1)} + \rho_2 E\nu_n^{(2)}.$$

From (9.7) we arrive at

$$(9.8) \quad E\nu_n^{(2)} = \frac{1}{1 - \rho_2} - \frac{1 - \rho_1}{1 - \rho_2} E\nu_n^{(1)},$$

which expresses $E\nu_n^{(2)}$ in terms of $E\nu_n^{(1)}$. For example, if $\rho_1 = 1$ then for any $n \geq 0$ we obtain

$$(9.9) \quad E\nu_n^{(2)} = \frac{1}{1 - \rho_2}.$$

That is in the case $\rho_1 = 1$, the value $E\nu_n^{(2)}$ is the same for all $n \geq 0$. This property is established in [5] together the aforementioned property of losses in $M/GI/1/n$ queues (see relation (2.10)).

Similarly, from (9.5) and (9.8) we obtain:

$$(9.10) \quad ET_n^{(2)} = \frac{1}{\mu(1-\rho_2)} - \frac{1-\rho_1}{\mu(1-\rho_2)} ET_n^{(1)}.$$

From equations (9.8) and (9.10) one can obtain the stationary probabilities p_1 and p_2 . By applying first renewal reward theorem (see e.g. [68]) and then (9.6) and (9.8), for p_1 and p_2 we obtain:

$$(9.11) \quad p_1 = \frac{EI_n}{ET_n^{(1)} + ET_n^{(2)} + EI_n} = \frac{1}{E\nu_n^{(1)} + E\nu_n^{(2)}} = \frac{1-\rho_2}{1 - (\rho_1 - \rho_2)E\nu_n^{(1)}},$$

$$(9.12) \quad p_2 = \frac{ET_n^{(2)}}{ET_n^{(1)} + ET_n^{(2)} + EI_n} = \frac{\rho_2 E\nu_n^{(2)}}{E\nu_n^{(1)} + E\nu_n^{(2)}} = \frac{\rho_2 + \rho_2(\rho_1 - 1)E\nu_n^{(1)}}{1 + (\rho_1 - \rho_2)E\nu_n^{(1)}}.$$

Thus, both probabilities p_1 and p_2 are expressed via $E\nu_n^{(1)}$. Our next arguments are based on sample path arguments and the property of the lack of memory of exponential distribution. Using all of this, one concludes that the random variable $\nu_n^{(1)}$ coincides in distribution with the number of customers served during a busy period of the $M/GI/1/n$ queueing system. This characteristics has been discussed in Section 2. According to Wald's equation we have the representation similar to (2.1). Specifically,

$$(9.13) \quad E\nu_n^{(1)} = \sum_{j=0}^n E(\nu_{n-j+1}^{(1)}) \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dB_1(x),$$

where $\nu_k^{(1)}$, $k = 0, 1, \dots, n-1$, are the numbers of served customers during a busy period associated with similar queueing systems (i.e. having the same rate λ of Poisson input and the same probability distribution functions $B_1(x)$ and $B_2(x)$ of the corresponding service times) but only defined by parameters k distinguished of n . As $n \rightarrow \infty$, the asymptotic behavior of $E\nu_n^{(1)}$ can be obtained from Takács' theorem. In turn, the stationary probabilities p_1 and p_2 are expressed via $E\nu_n^{(1)}$, and therefore we have the following asymptotic theorem.

Theorem 9.1. *If $\rho_1 < 1$, then*

$$(9.14) \quad \lim_{n \rightarrow \infty} p_1(n) = 1 - \rho_1,$$

$$(9.15) \quad \lim_{n \rightarrow \infty} p_2(n) = 0.$$

If $\rho_1 = 1$, then

$$(9.16) \quad \lim_{n \rightarrow \infty} np_1(n) = \frac{\rho_{1,2}}{2},$$

$$(9.17) \quad \lim_{n \rightarrow \infty} np_2(n) = \frac{\rho_2}{1 - \rho_2} \cdot \frac{\rho_{1,2}}{2}.$$

If $\rho_1 > 1$, then

$$(9.18) \quad \lim_{n \rightarrow \infty} \frac{p_1(n)}{\varphi^n} = \frac{(1 - \rho_2)[1 + \lambda \widehat{B}_1'(\lambda - \lambda\varphi)]}{\rho_1 - \rho_2},$$

where $\widehat{B}_1(s)$ is the Laplace-Stieltjes transform of $B_1(x)$, and φ is the least positive root of equation $z = \widehat{B}_1(\lambda - \lambda z)$, and

$$(9.19) \quad \lim_{n \rightarrow \infty} p_2(n) = \frac{\rho_2(\rho_1 - 1)}{\rho_1 - \rho_2}.$$

Let us discuss a relation of Theorem 9.1 to optimal control problem. Under the assumption that $\rho_1 < 1$, relations (9.14) and (9.15) enables us to conclude as follows. The probability p_1 is positive in limit, while the probability p_2 vanishes. Therefore, in the case $\rho_1 < 1$ we have $J \approx (1 - \rho_1)J_1$, so J increases to infinity being asymptotically equivalent to $(1 - \rho_1)j_1 n$. The similar conclusion follows from (9.16) and (9.17) under the assumption $\rho_1 > 1$. In this case the probability p_1 vanishes, while the probability p_2 tends to the positive limit $\frac{\rho_2(\rho_1 - 1)}{(\rho_1 - \rho_2)}$. This enables us to conclude that J increases to infinity together with n increasing to infinity being asymptotically equivalent to $\frac{\rho_2(\rho_1 - 1)}{(\rho_1 - \rho_2)} j_2 n$. In the case $\rho_1 = 1$ both p_1 and p_2 vanish at rate $O(\frac{1}{n})$, and J therefore converges to the limit as $n \rightarrow \infty$. Thus, $\rho_1 = 1$ is a possible solution to the control problem, while the cases $\rho_1 < 1$ and $\rho_1 > 1$ are irrelevant. In the case $\rho_1 = 1$ we have as follows:

$$(9.20) \quad \lim_{n \rightarrow \infty} J(n) = j_1 \frac{\rho_{1,2}}{2} + j_2 \frac{\rho_2}{1 - \rho_2} \cdot \frac{\rho_{1,2}}{2}.$$

According to the result of (9.20), the class of possible solutions to the control problem can be described by the following two cases, in both of which $\delta \rightarrow 0$ ($\delta > 0$) and $n \rightarrow \infty$: (i) $\rho_1 = 1 + \delta$; (ii) $\rho_1 = 1 - \delta$.

In case (i) we have the following two theorems.

Theorem 9.2. *Assume that $\rho_1 = 1 + \delta$, $\delta > 0$, and that $n\delta \rightarrow C > 0$ as $\delta \rightarrow 0$ and $n \rightarrow \infty$. Assume that $\rho_{1,3} = \rho_{1,3}(n)$ is a bounded sequence and that the limit $\tilde{\rho}_{1,2} = \lim_{n \rightarrow \infty} \rho_{1,2}(n)$ exists. Then,*

$$(9.21) \quad p_1 = \frac{\delta}{e^{2C/\tilde{\rho}_{1,2}} - 1} + o(\delta),$$

$$(9.22) \quad p_2 = \frac{\delta \rho_2 e^{2C/\tilde{\rho}_{1,2}}}{(1 - \rho_2)(e^{2C/\tilde{\rho}_{1,2}} - 1)} + o(\delta).$$

Theorem 9.3. *Under the conditions of Theorem 9.2, assume that $C = 0$. Then,*

$$(9.23) \quad \lim_{n \rightarrow \infty} np_1(n) = \frac{\rho_{1,2}}{2},$$

$$(9.24) \quad \lim_{n \rightarrow \infty} np_2(n) = \frac{\rho_2}{1 - \rho_2} \cdot \frac{\rho_{1,2}}{2}.$$

The proof of these two theorems is based on the above expansions given by (2.13) and (2.14), and thus the proof is similar to the proof of aforementioned Theorems 2.5 and 2.6. For details of the proofs see [12].

In case (ii) we have the following two theorems.

Theorem 9.4. *Assume that $\rho_1 = 1 - \delta$, $\delta > 0$, and that $n\delta \rightarrow C > 0$ as $\delta \rightarrow 0$ and $n \rightarrow \infty$. Assume that $\rho_{1,3} = \rho_{1,3}(n)$ is a bounded sequence, and that the limit $\tilde{\rho}_{1,2} = \lim_{n \rightarrow \infty} \rho_{1,2}(n)$ exists. Then,*

$$(9.25) \quad p_1 = \delta e^{\tilde{\rho}_{1,2}/2C} + o(\delta),$$

$$(9.26) \quad p_2 = \delta \frac{\rho_2}{1 - \rho_2} (e^{\tilde{\rho}_{1,2}} - 1) + o(\delta).$$

Theorem 9.5. *Under the conditions of Theorem 9.4, assume that $C = 0$. Then we obtain (9.23) and (9.24).*

Theorems 9.4 and 9.5 do not follow directly from other theorems as this was in the case of Theorem 9.2, which is derived from 9.1 by using appropriate asymptotic expansions. The proof of Theorem 9.4 is based on Tauberian theorem of Hardy and Littlewood which can be found in many sources (e.g. [65], [71], [73], [75], p. 203 and [89]). Below we give the proof of Theorem 9.4 based on the aforementioned Tauberian theorem.

We have the representation

$$\sum_{n=0}^{\infty} E\nu_n^{(1)} z^n = \frac{\widehat{B}_1(\lambda - \lambda z)}{\widehat{B}_1(\lambda - \lambda z) - z},$$

which is the consequence of (1.3). The sequence $\{E\nu_n^{(1)}\}$ is increasing, and, for $\rho_1 = 1$, from the aforementioned Tauberian theorem of Hardy and Littlewood we have

$$\lim_{n \rightarrow \infty} \frac{E\nu_n^{(1)}}{n} = \lim_{z \uparrow 1} (1 - z)^2 \frac{\widehat{B}_1(\lambda - \lambda z)}{\widehat{B}_1(\lambda - \lambda z) - z}.$$

In the case $\rho_1 = 1 - \delta$ and $\delta n \rightarrow C$ as $n \rightarrow \infty$ and $\delta \rightarrow 0$, according to the same Tauberian theorem of Hardy and Littlewood, the asymptotic behavior of $E\nu_n^{(1)}$ can be found from the asymptotic expansion of

$$(9.27) \quad (1 - z) \frac{\widehat{B}(\lambda - \lambda z)}{\widehat{B}(\lambda - \lambda z) - z},$$

as $z \uparrow 1$. Expanding the denominator of (9.27) to the Taylor series, we obtain:

$$\begin{aligned} \frac{1 - z}{\widehat{B}(\lambda - \lambda z) - z} &= \frac{1 - z}{1 - z - \rho_1(1 - z) + (\widetilde{\rho}_{1,2}/2)(1 - z)^2 + O((1 - z)^3)} \\ &= \frac{1}{\delta + (\widetilde{\rho}_{1,2}/2)(1 - z) + O((1 - z)^2)} \\ (9.28) \quad &= \frac{1}{\delta[1 + (\widetilde{\rho}_{1,2}/2\delta)(1 - z) + O((1 - z)^2)]} \\ &= \frac{1}{\delta \exp((\widetilde{\rho}_{1,2}/2\delta)(1 - z))} (1 + o(1)). \end{aligned}$$

Therefore, assuming that $z = (n - 1)/n \rightarrow 1$ as $n \rightarrow \infty$, from (9.28) we have:

$$(9.29) \quad E\nu_n^{(1)} = \frac{1}{\delta e^{\widetilde{\rho}_{1,2}/2C}} (1 + o(1)).$$

Substituting (9.29) into (9.11) and (9.12) we arrive at the desired statement of Theorem 9.4. The proof of Theorem 9.5 is similarly based on the above expansion.

Above Theorems 9.2 - 9.5 enable us to solve the control problem. We have the following limiting relation:

$$\begin{aligned} (9.30) \quad \lim_{n \rightarrow \infty} J(n) &= \lim_{n \rightarrow \infty} [p_1(n)J_1(n) + p_2(n)J_2(n)] \\ &= j_1 \lim_{n \rightarrow \infty} np_1(n) + j_2 \lim_{n \rightarrow \infty} np_2(n). \end{aligned}$$

According to the cases (i) and (ii) we have two corresponding functionals. Specifically, substituting (9.21) and (9.22) into the right-hand side of (9.30) and taking

into account that $n\delta \rightarrow C$ we obtain:

$$(9.31) \quad J^{upper} = C \left[j_1 \frac{1}{e^{2C/\tilde{\rho}_{1,2}} - 1} + j_2 \frac{\rho_2 e^{2C/\tilde{\rho}_{1,2}}}{(1 - \rho_2)(e^{2C/\tilde{\rho}_{1,2}} - 1)} \right].$$

Next, substituting (9.25) and (9.26) into the right-hand side of (9.30) and taking into account that $n\delta \rightarrow C$ we obtain:

$$(9.32) \quad J^{lower} = C \left[j_1 e^{\tilde{\rho}_{1,2}/2C} + j_2 \frac{\rho_2}{1 - \rho_2} (e^{\tilde{\rho}_{1,2}/2C} - 1) \right].$$

An elementary analysis of functionals (9.31) and (9.32) shows, that the minimum of the both of them is achieved under $C = 0$ if and only if

$$(9.33) \quad j_1 = j_2 \frac{\rho_2}{1 - \rho_2}.$$

More detailed analysis of these functionals (9.31) and (9.32) (see [12]) leads to the following solution to the control problem.

Theorem 9.6. *If the parameters λ and ρ_2 are given, then the optimal solution to the control problem is as follows.*

If

$$j_1 = j_2 \frac{\rho_2}{1 - \rho_2},$$

then the optimal solution to the control problem is achieved for $\rho_1 = 1$.

If

$$j_1 > j_2 \frac{\rho_2}{1 - \rho_2},$$

then the optimal solution to the control problem is a minimization of the functional J^{upper} . The optimal solution is achieved for $\rho_1 = 1 + \delta$, where $\delta(n)$ is a small positive parameter and $n\delta(n) \rightarrow C$, the nonnegative parameter minimizing (9.31).

If

$$j_1 < j_2 \frac{\rho_2}{1 - \rho_2},$$

then the optimal solution to the control problem is a minimization of the functional J^{lower} . The optimal solution is achieved for $\rho_1 = 1 - \delta$, where $\delta(n)$ is a small positive parameter and $n\delta(n) \rightarrow C$, the nonnegative parameter minimizing (9.32).

There is a large number of papers in the dam literature concerning different optimal control problem of water of storage resources. To indicate only a few of them we refer [1], [2], [20], [40], [51], [52] and [96]. However, the optimal control problem of dams and its solution, which is discussed in this section of the paper, specifically differs from all of the earlier considerations known from the literature. In the next section we discuss a more extended control problem of large dams.

10. OPTIMAL POLICIES OF USING WATER IN LARGE DAMS: AN EXTENDED MODEL

In the previous section we found optimal solution to the control problem of minimization the objective function

$$J = p_1 J_1 + p_2 J_2,$$

where p_1 and p_2 are stationary probabilities of passage across the lower and upper levels of a dam, i.e.

$$\begin{aligned} p_1 &= \lim_{t \rightarrow \infty} P\{L_t = L^{lower}\}, \\ p_2 &= \lim_{t \rightarrow \infty} P\{L_t > L^{upper}\}, \end{aligned}$$

L_t denotes the water level in time t , and J_1, J_2 are the cost functions having the form: $J_1 = j_1 n$ and $J_2 = j_2 n$. The extended problem considered in this section is the problem of minimization of the functional

$$(10.1) \quad J = p_1 J_1 + p_2 J_2 + \sum_{i=L^{lower}+1}^{L^{upper}} c_i q_i,$$

where

$$(10.2) \quad q_i = \lim_{t \rightarrow \infty} P\{L_t = L^{lower} + i\}, \quad i = 1, 2, \dots, n,$$

and c_i is the water cost when the level of dam is equal to i . The costs c_i are assumed to be decreasing in i , i.e. $c_{i+1} \leq c_i$ for all $1 \leq i \leq n-1$. (The water is cheaper when the dam level is higher.)

In queueing formulation the level L^{lower} is equated with an empty queue. The queueing formulation of the dam model is given in the previous section. All of the assumptions for that state-dependent queueing system remains the same.

It was shown in the previous section that the probabilities p_1 and p_2 are expressed via $E\nu_n^{(1)}$ (relations (9.11) and (9.12)), and the asymptotic representations of p_1 and p_2 is derived from the asymptotic formula of $E\nu_n^{(1)}$. The stationary probabilities q_i , $i = 1, 2, \dots, n$ can be obtained from renewal arguments (see e.g. Ross [68]). Namely, for $i = 1, 2, \dots, n$ we have:

$$(10.3) \quad q_i = \frac{ET_i^{(1)} - ET_{i-1}^{(1)}}{ET_n + EI_n}.$$

($ET_i^{(1)}$ means the expectation of the total time that the customers are served by probability distribution function $B_1(x)$ during a busy period of the state-dependent system, which is distinguished from the described system only by parameter i given instead of the original parameter n . According to sample-path arguments and the property of the lack of memory of exponential distribution (these arguments are given in the previous section), the random variable $T_i^{(1)}$ coincides in distribution with a busy period of the $M/GI/1/i$ queueing system.) The probabilities q_i , $i = 1, 2, \dots, n$, given by (10.3), can be also rewritten

$$(10.4) \quad q_i = \rho_1 \frac{E\nu_i^{(1)} - E\nu_{i-1}^{(1)}}{E\nu_n}.$$

The equivalence of (10.3) and (10.4) follows easily from the equations $E\nu_i^{(1)} = \mu_1 ET_i^{(1)}$ for all $i = 0, 1, \dots, n$, and $E\nu_i^{(1)} = \lambda ET_n + \lambda EI_n$, which are Wald's equations.

Representation (10.4) can also be rewritten in other forms, which are more convenient for our purposes. Recall that (see relation (9.8))

$$E\nu_n^{(2)} = \frac{1}{1 - \rho_2} - \frac{1 - \rho_1}{1 - \rho_2} E\nu_n^{(1)},$$

and therefore, taking into account that $E\nu_n = E\nu_n^{(1)} + E\nu_n^{(2)}$, we also have:

$$(10.5) \quad E\nu_n = \frac{1}{1 - \rho_2} + \frac{\rho_1 - \rho_2}{1 - \rho_2} E\nu_n^{(1)}.$$

Then elementary substitution of (10.5) into (10.4) gives us

$$(10.6) \quad q_i = \frac{\rho_1(1 - \rho_2)}{1 + (\rho_1 - \rho_2)E\nu_n^{(1)}} \left(E\nu_i^{(1)} - E\nu_{i-1}^{(1)} \right), \quad i = 1, 2, \dots, n.$$

Comparison with (9.11) enables us to rewrite (10.6) in the other form:

$$(10.7) \quad q_i = \rho_1 p_1 \left(E\nu_i^{(1)} - E\nu_{i-1}^{(1)} \right), \quad i = 1, 2, \dots, n.$$

Let us now discuss the different cases as (i) $\rho_1 = 1$, and for sufficiently small $\delta > 0$ (ii) $\rho_1 = 1 + \delta$, and (iii) $\rho_1 = 1 - \delta$.

We start from case (i). In this case we have the following result.

Theorem 10.1. *If $\rho_1 = 1$ and $\rho_2 < \infty$, then for all $i = 1, 2, \dots, n$*

$$(10.8) \quad \lim_{n \rightarrow \infty} n q_{n-i} = 1.$$

The proof of this theorem is based on application Tauberian Theorem 1.3. Namely, from that Tauberian theorem we have as follows.

Lemma 10.2. *Under the conditions of Theorem 10.1 for any $j \geq 0$ we have*

$$(10.9) \quad E\nu_{n-j}^{(1)} - E\nu_{n-j-1}^{(1)} = \frac{2}{\rho_{1,2}} + o(1),$$

as $n \rightarrow \infty$.

Then, the statement of Theorem 10.1 follows by application of Lemma 10.2, i.e. by substitution (10.9) into (10.7) and consequent application of relation (9.16) of Theorem 9.1. Notice, that just Tauberian theorem 1.3 is applied here, since the appropriate statement of Takács' theorem is not enough in order to prove the required statement of Theorem 10.1.

In turn, Theorem 10.1 leads to the following result.

Proposition 10.3. *Under the conditions of Theorem 10.1 we have*

$$(10.10) \quad \lim_{n \rightarrow \infty} J(n) = j_1 \frac{\rho_{1,2}}{2} + j_2 \frac{\rho_2}{1 - \rho_2} \cdot \frac{\rho_{1,2}}{2} + c^*,$$

where

$$c^* = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n c_i.$$

Notice, that the only difference between (9.20) and representation (10.10) is in the presence of the term c^* in (10.10).

In case (ii) we have the following theorem.

Theorem 10.4. *Assume that $\rho_1 = 1 + \delta$, $\delta > 0$ and $n\delta \rightarrow C > 0$ as $\delta \rightarrow 0$ and $n \rightarrow \infty$. Assume also that $\rho_{1,3}(n)$ is a bounded sequence and there exists $\tilde{\rho}_{1,2} = \lim_{n \rightarrow \infty} \rho_{1,2}(n)$. Then, for any $j \geq 0$*

$$(10.11) \quad q_{n-j} = \frac{e^{2C/\tilde{\rho}_{1,2}}}{e^{2C/\tilde{\rho}_{1,2}} - 1} \left(1 - \frac{2\delta}{\tilde{\rho}_{1,2}} \right)^j \frac{2\delta}{\tilde{\rho}_{1,2}} + o(\delta).$$

The proof of this theorem is based on the expansion

$$\mathbb{E}\nu_{n-j}^{(1)} = \frac{\varphi^j}{\varphi^n [1 + \lambda \widehat{B}'_1(\lambda - \lambda\varphi)]} + \frac{1}{1 - \rho_1} + o(1),$$

which is the consequence of an application of Takács' theorem to recurrence relation (9.13), as well as expansions (2.13) and (2.14).

From Theorem 10.4 we have the following result.

Proposition 10.5. *Under the assumptions of Theorem 10.4 let us denote the objective function J by J^{upper} . We have the representation*

$$(10.12) \quad J^{upper} = C \left[j_1 \frac{1}{e^{2C/\tilde{\rho}_{1,2}} - 1} + j_2 \frac{\rho_2 e^{2C/\tilde{\rho}_{1,2}}}{(1 - \rho_2)(e^{2C/\tilde{\rho}_{1,2}} - 1)} \right] + c^{upper},$$

where

$$(10.13) \quad c^{upper} = \frac{2C}{\tilde{\rho}_{1,2}} \cdot \frac{e^{2C/\tilde{\rho}_{1,2}}}{e^{2C/\tilde{\rho}_{1,2}} - 1} \lim_{n \rightarrow \infty} \widehat{C}_n \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n} \right),$$

and $\widehat{C}_n(z) = \sum_{j=0}^{n-1} c_{n-j} z^j$ is a backward generating cost function.

For details of the proof see [13].

Notice, that the term

$$C \left[j_1 \frac{1}{e^{2C/\tilde{\rho}_{1,2}} - 1} + j_2 \frac{\rho_2 e^{2C/\tilde{\rho}_{1,2}}}{(1 - \rho_2)(e^{2C/\tilde{\rho}_{1,2}} - 1)} \right]$$

is the value of the function J^{upper} for the model where the water costs have not been taken into account (see relation (9.31)). So, the function c^{upper} given by (10.13) is a new element for the function J^{upper} .

Case (iii) is more delicate. The additional assumption that required here is that the class of probability distributions $\{B_1(x)\}$ is such that there exists a unique root $\tau > 1$ of the equation

$$(10.14) \quad z = \widehat{B}(\lambda - \lambda z),$$

and there also exists the derivative $\widehat{B}'(\lambda - \lambda\tau)$. In general, under the assumption that $\rho_1 < 1$ the root of equation (10.14) not necessarily exists. Such type of condition has been considered by Willmot [91] in order to obtain the asymptotic behavior for the probability of high-level queue-length in stationary $M/GI/1$ queueing systems. Let $q_i[M/GI/1]$, $i = 0, 1, \dots$ denote the stationary queue-length probabilities. Willmot [91] showed that

$$(10.15) \quad q_i[M/GI/1] = \frac{(1 - \rho_1)(1 - \tau)}{\tau^i [1 + \lambda \widehat{B}'(\lambda - \lambda\tau)]} [1 + o(1)],$$

as $i \rightarrow \infty$. On the other hand, there is the following representation:

$$(10.16) \quad q_i[M/GI/1] = (1 - \rho_1) \left(\mathbb{E}\nu_i^{(1)} - \mathbb{E}\nu_{i-1}^{(1)} \right), \quad i = 1, 2, \dots,$$

which agrees with the well-known Pollaczek-Khintchin formula (e.g. Takács [74], p. 242).

From (10.15) and (10.16) for any $j \geq 0$ we have the following asymptotic proportion:

$$(10.17) \quad \frac{\mathbb{E}\nu_{n-j} - \mathbb{E}\nu_{n-j-1}}{\mathbb{E}\nu_n^{(1)} - \mathbb{E}\nu_{n-1}^{(1)}} = \tau^j [1 + o(1)].$$

In order to formulate and prove a theorem on asymptotic behavior of stationary probabilities q_i for case (iii) we assume that the class of probability distributions $\{B_1(x)\}$ is as follows. Under the assumption that $\rho_1 = 1 - \delta$, $\delta > 0$, and $\delta \rightarrow 0$ and $n \rightarrow \infty$, we assume that there exists the value $\epsilon_0 > 0$ small enough (proportionally to δ) such that for all $0 \leq \epsilon \leq \epsilon_0$ the family of probability distributions $B_{1,\epsilon}(x)$, provided now by parameter ϵ , satisfies the following condition: its Laplace-Stieltjes transform $\hat{B}_{1,\epsilon}(z)$ is an analytic function in a small neighborhood of zero and

$$(10.18) \quad \hat{B}'_{1,\epsilon}(\lambda\epsilon) < \infty.$$

We have the following theorem.

Theorem 10.6. *Assume that the class of probability distributions $B_{1,\epsilon}(x)$ is defined according to the above convention and satisfies (10.18). Assume that $\rho_1 = 1 - \delta$, $\delta > 0$, and $n\delta \rightarrow C > 0$ as $\delta \rightarrow 0$ and $n \rightarrow \infty$. Assume that $\rho_{1,3}(n)$ is a bounded sequence and there exists $\tilde{\rho}_2 = \lim_{n \rightarrow \infty} \rho_{1,2}(n)$. Then,*

$$(10.19) \quad q_{n-j} = \frac{2\delta}{\tilde{\rho}_{1,2}} \cdot \frac{1}{e^{2C/\tilde{\rho}_{1,2}} - 1} \left(1 + \frac{2\delta}{\tilde{\rho}_{1,2}}\right)^j [1 + o(1)]$$

for any $j \geq 0$.

The proof of this theorem uses the expansion

$$\tau = 1 + \frac{2\delta}{\tilde{\rho}_{1,2}} + O(\delta^2),$$

which is similar to the expansion of (2.13) (for more details see [13]).

From this theorem we arrive at the following proposition.

Proposition 10.7. *Under the assumptions of Theorem 10.6 denote the objective function J by J^{lower} . We have the following representation:*

$$(10.20) \quad J^{lower} = C \left[j_1 e^{\tilde{\rho}_{1,2}/2C} + j_2 \frac{\rho_2}{1 - \rho_2} \left(e^{\tilde{\rho}_{1,2}/2C} - 1 \right) \right] + c^{lower},$$

where

$$(10.21) \quad c^{lower} = \frac{2C}{\tilde{\rho}_{1,2}} \cdot \frac{1}{e^{2C/\tilde{\rho}_{1,2}} - 1} \lim_{n \rightarrow \infty} \frac{1}{n} \hat{C}_n \left(1 + \frac{2C}{\tilde{\rho}_{1,2}n} \right),$$

and $\hat{C}(z) = \sum_{j=0}^{n-1} c_{n-j} z^j$ is a backward generating cost function.

For details of the proof see [13].

Notice, that the term

$$C \left[j_1 e^{\tilde{\rho}_{1,2}/2C} + j_2 \frac{\rho_2}{1 - \rho_2} \left(e^{\tilde{\rho}_{1,2}/2C} - 1 \right) \right]$$

is the value of the function J^{lower} for the model where the water costs have not been taken into account (see relation (9.32)). So, the function c^{lower} given by (10.21) is a new element for the function J^{lower} .

Representations (10.13) and (10.21) are not convenient, and for the purpose of the further analysis we provide other representations.

Introduce the following functions:

$$(10.22) \quad \psi(C) = \lim_{n \rightarrow \infty} \frac{\sum_{j=0}^{n-1} c_{n-j} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n} \right)^j}{\sum_{j=0}^{n-1} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n} \right)^j},$$

$$(10.23) \quad \eta(C) = \lim_{n \rightarrow \infty} \frac{\sum_{j=0}^{n-1} c_{n-j} \left(1 + \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j}{\sum_{j=0}^{n-1} \left(1 + \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j}.$$

Since c_i is a bounded sequence, then the both limits of (10.22) and (10.23) do exist. Immediate algebraic calculations show that $c^{upper} = \psi(C)$ and $c^{lower} = \eta(C)$ if one substitutes the exact representation for the corresponding limits:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j = \frac{\tilde{\rho}_{1,2}}{2C} \left(1 - e^{-2C/\tilde{\rho}_{1,2}}\right),$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 + \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j = \frac{\tilde{\rho}_{1,2}}{2C} \left(e^{2C/\tilde{\rho}_{1,2}} - 1\right).$$

The functions $\psi(C)$ and $\eta(C)$ satisfy the following properties. The function $\psi(C)$ is a decreasing function, and its maximum is $\psi(0) = c^*$. The function $\eta(C)$ is an increasing function, and its minimum is $\eta(0) = c^*$. Recall that $c^* = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n c_i$.

The proof of these properties is based on the following properties of numerical series (see Hardy, Littlewood and Polya [45] or Marschall and Olkin [56]). Let $\{a_n\}$ and $\{b_n\}$ be arbitrary sequences of numbers. If one of them is increasing but another is decreasing, then for any finite sum

$$(10.24) \quad \sum_{n=1}^l a_n b_n \leq \frac{1}{l} \sum_{n=1}^l a_n \sum_{n=1}^l b_n.$$

If both of these sequences are increasing or decreasing, then

$$(10.25) \quad \sum_{n=1}^l a_n b_n \geq \frac{1}{l} \sum_{n=1}^l a_n \sum_{n=1}^l b_n.$$

Rewrite (10.22) and (10.23) as follows:

$$(10.26) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} c_{n-j} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j = \psi(C) \lim_{n \rightarrow \infty} \sum_{j=1}^{n-1} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j,$$

and

$$(10.27) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} c_{n-j} \left(1 + \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j = \eta(C) \lim_{n \rightarrow \infty} \sum_{j=1}^{n-1} \left(1 + \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j.$$

Applying inequality (10.24) to the left-hand side of (10.26) and letting $n \rightarrow \infty$, we obtain:

$$(10.28) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} c_{n-j} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} c_{n-j} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j \\ & = \psi(0) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j. \end{aligned}$$

Comparing (10.26) with (10.28) we arrive at

$$\psi(0) = c^* \geq \psi(C),$$

i.e. $\psi(0) = c^*$ is the maximum value of $\psi(C)$.

To show that $\psi(C)$ is a decreasing function, we are actually to show that for any nonnegative $C_1 \leq C$ we have $\psi(C) \leq \psi(C_1)$. The proof of this fact is based on the following asymptotic relation. For small positive δ_1 and δ_2 we have $1 - \delta_1 - \delta_2 = (1 - \delta_1)(1 - \delta_2) + O(\delta_1\delta_2)$. In out terms, this means that for $C_1 < C$ we have:

$$1 - \frac{2C}{\tilde{\rho}_{1,2}n} = \left(1 - \frac{2C_1}{\tilde{\rho}_{1,2}n}\right) \left(1 - \frac{2C - 2C_1}{\tilde{\rho}_{1,2}n}\right) + O\left(\frac{1}{n^2}\right).$$

Therefore, similarly to (10.28) for the left-hand side of (10.26) we have:⁵

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} c_{n-j} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} c_{n-j} \left(1 - \frac{2C_1}{\tilde{\rho}_{1,2}n}\right)^j \left(1 - \frac{2C - 2C_1}{\tilde{\rho}_{1,2}n}\right)^j \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} c_{n-j} \left(1 - \frac{2C_1}{\tilde{\rho}_{1,2}n}\right)^j \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 - \frac{2C - 2C_1}{\tilde{\rho}_{1,2}n}\right)^j \\ &= \psi(C_1) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 - \frac{2C_1}{\tilde{\rho}_{1,2}n}\right)^j \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 - \frac{2C - 2C_1}{\tilde{\rho}_{1,2}n}\right)^j. \end{aligned} \tag{10.29}$$

On the other hand, by using inequality (10.25) for the right-hand side of (10.26) we have:

$$\begin{aligned} & \psi(C) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 - \frac{2C}{\tilde{\rho}_{1,2}n}\right)^j \\ &= \psi(C) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 - \frac{2C_1}{\tilde{\rho}_{1,2}n}\right)^j \left(1 - \frac{2C - 2C_1}{\tilde{\rho}_{1,2}n}\right)^j \\ &\geq \psi(C) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 - \frac{2C_1}{\tilde{\rho}_{1,2}n}\right)^j \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(1 - \frac{2C - 2C_1}{\tilde{\rho}_{1,2}n}\right)^j. \end{aligned} \tag{10.30}$$

Hence, the above monotonicity property follows from (10.29) and (10.30). Notice, that the inequality in (10.30) is in fact the equality.

Similarly, on the base of relation (10.27) one can show the monotonicity of the other function $\eta(C)$.

All of these properties enables us to formulate the following general result.

Theorem 10.8. *The solution to the control problem is $\rho_1 = 1$ if and only if the minimum of the both functionals J^{upper} and J^{lower} is achieved for $C = 0$. In this case, the minimum of objective function J is given by (10.10). Otherwise, there can be one of the following two cases for the solution to the control problem.*

⁵This is a very short version of the original proof given in [13]. If $c_i \equiv c$, i.e. all coefficients are equal to the same constant, then the inequality in (10.29) becomes the equality. For the following technical details associated with this property see [13].

(1) If the minimum of J^{upper} is achieved for $C = 0$, then the minimum of J^{lower} must be achieved for some positive value $C = \underline{C}$. Then the solution to the control problem is achieved for $\rho_1 = 1 - \delta$, $\delta > 0$, such that $\delta n \rightarrow \underline{C}$ as $n \rightarrow \infty$.

(2) If the minimum of J^{lower} is achieved for $C = 0$, then the minimum of J^{upper} must be achieved for some positive value $C = \overline{C}$. Then the solution to the control problem is achieved for $\rho_1 = 1 + \delta$, $\delta > 0$, such that $\delta n \rightarrow \overline{C}$ as $n \rightarrow \infty$.

The minimum of the functionals J^{upper} and J^{lower} can be found by their differentiating in C and then equating these derivatives to zero.

From Theorem 10.8 we have the following property of the optimal control.

Corollary 10.9. *The solution to the control problem can be $\rho_1 = 1$ only in the case where*

$$(10.31) \quad j_1 \leq j_2 \frac{\rho_2}{1 - \rho_2}.$$

The equality in relation (10.31) is achieved only for $c_i \equiv c$, $i = 1, 2, \dots, n$, where c is an any positive constant.

Paper [13] discusses the case where the costs c_i have a linear structure, and provides numerical analysis of this case. Specifically, [13] finds numerically the relation between j_1 and j_2 , under which the optimal solution to the control problem is $\rho_1 = 1$.

11. OPTIMAL POLICIES OF USING WATER IN LARGE DAMS: FUTURE RESEARCH PROBLEMS

In Sect. 9 and 10, optimal policies of using water in large dams are studied. Both these models are the same queueing systems with specified dependence of a service on queue-length, and in both of them the input stream is Poisson. The only difference is in the criteria of minimization. The minimization criteria in the simple model is based on penalties for passage across the lower and higher level. The minimization criteria of the extended model takes also into account the water costs depending on the level of water in the dam.

However, the assumption that input stream is Poisson is restrictive. So, the following extension of these models towards their clearer application in practice lies in an assumption of a wider class of input process than ordinary Poisson. In this section we discuss a possible way of analysis in the case when the input stream is a compound Poisson process.

Let t_1, t_2, \dots be the moment of arrivals of units at the system, and let $\kappa_1, \kappa_2, \dots$ be independent and identically distributed integer random variables characterizing the corresponding lengths of these units. As in Sect. 7 it is reasonable to assume that

$$P\{\kappa^{lower} \leq \kappa_i \leq \kappa^{upper}\} = 1$$

(see rel. (7.1)).

Let us recall the meaning of parameter n in the dam models of Sect. 9 and 10. That parameter n is defined in the following formulation of queueing system. Consider a single-server queueing system where the arrival flow of customers is Poisson with rate λ and the service time of a customer depends upon queue-length as follows. If, at the moment the customer's service beginning, the number of customers in the system is not greater than n , then the service time of this customer has

the probability distribution $B_1(x)$. Otherwise (if there are more than n customers in the system at the moment of the customer's service beginning), the probability distribution function of the service time of this customer is $B_2(x)$.

In our case customers arrive by batches $\kappa_1, \kappa_2, \dots$, so as in Sect. 7 it is reasonable to define the random parameter $\zeta = \zeta(n)$:

$$\zeta = \sup \left\{ m : \sum_{i=1}^m \kappa_i \leq n \right\},$$

There are two fixed values ζ^{lower} and ζ^{upper} depending on n and

$$P\{\zeta^{lower} \leq \kappa_i \leq \zeta^{upper}\} = 1.$$

Thus, the following construction, similar to the construction that used in Sect. 7, holds in this case as well. Specifically, Definition 7.1 and the following application of the formulae for the total expectation for characteristic of the system with random parameter ζ are the same as above in Sect. 7. Then the definition of random variables $T_\zeta^{(1)}, T_\zeta^{(2)}, \nu_\zeta^{(1)}, \nu_\zeta^{(2)}$ with random parameter ζ is similar to the definition of the corresponding random variables $T_n^{(1)}, T_n^{(2)}, \nu_n^{(1)}, \nu_n^{(2)}$ given in Sect. 9 and 10.

There are also equalities allying $E\nu_{\zeta(n)}^{(1)}$ with $E\nu_{\zeta(n)}^{(2)}$ and $ET_{\zeta(n)}^{(1)}$ with $ET_{\zeta(n)}^{(2)}$ as

$$E\nu_{\zeta(n)}^{(2)} = a_{1,n}E\nu_{\zeta(n)}^{(1)} + a_{2,n}$$

and

$$ET_{\zeta(n)}^{(2)} = b_{1,n}ET_{\zeta(n)}^{(1)} + b_{2,n}.$$

The sequences $a_{1,n}, a_{2,n}, b_{1,n}$ and $b_{2,n}$ converge to the corresponding limits a_1, a_2, b_1 and b_2 . But the values of constants a_1, a_2, b_1 and b_2 are not the same as those given by (9.8) and (9.10). Special analysis based on renewal theory to find these constants is required.

Another difference between the analysis of the problem described in Sect. 10 and that analysis of the present problem is connected with the structures of water costs. In the case of the problem described in Sect. 10 the water costs are non-increasing. In the present formulation, where the length of arrival units are random, the water costs, being initially non-increasing with respect to levels of water, after reduction to the model with random parameter ζ become random variables. However, the expected values of these random costs satisfy the same property, they are non-increasing.

So, the statement on existence and uniqueness of an optimal strategy of water consumption is anticipated to be the same as in the earlier consideration in Sect. 10 where, however, the optimal solution itself and its structure can be different.

12. THE BUFFER MODEL WITH PRIORITIES

In this section we give the application of Takács' theorem for a specific buffer model with priorities. This section contains one of the results of [15] related to the effective bandwidth problem [15], and the model studied here is a particular case of more general models of [15]. The effective bandwidth problem was a "hot topic" of applications of queueing theory during the decade 1990-2000. The detailed review of the existence literature up to the publication time can be found in the paper of Berger and Whitt [21]. For other papers published later than aforementioned one see [22], [24], [34], [38], [39], [50], [53], [92], [93] and other papers.

Let $\mathcal{A}(t)$ be a renewal process of ordinary arrivals to telecommunication system having a large buffer of capacity N . Each of these arrivals belongs to the priority class k with positive probability $p^{(k)}$, $\sum_{k=1}^{\ell} p^{(k)} = 1$, where a smaller index corresponds to a higher priority. By thinning the renewal process $\mathcal{A}(t)$ we thus have ℓ independent renewal processes $\mathcal{A}^{(1)}(t), \mathcal{A}^{(2)}(t), \dots, \mathcal{A}^{(\ell)}(t)$, where the highest priority units are associated with the process $\mathcal{A}^{(1)}(t)$, and arrivals corresponding to $\mathcal{A}^{(i)}(t)$ have higher priority than arrivals corresponding to $\mathcal{A}^{(j)}(t)$ for $i < j$.

The departure process, $\mathcal{D}(t)$, is assumed to be a Poisson process with constant integer jumps $C \geq 1$. The process $\mathcal{D}(t)$ is common for all of units independently of their priorities.

Let $\mathcal{Q}^{(k)}(t)$ denote the buffer content in time t for units of priority k . All of the processes considered here are assumed to be right continuous, having left limits, and all of them are assumed to start at zero. Exceptions from these rules will be mentioned especially.

Let us now describe the priority rule. To simplify the explanation, let us first assume that the buffer is infinite. Then, for the highest priority units we have:

$$(12.1) \quad \mathcal{Q}^{(1)}(t) = \max\{0, \mathcal{Q}^{(1)}(t-) + \triangle \mathcal{A}^{(1)}(t) - \triangle \mathcal{D}(t)\},$$

where the triangle in (12.1) denotes the value of jump of the process in point t , i.e. $\triangle \mathcal{A}^{(1)}(t) = \mathcal{A}^{(1)}(t) - \mathcal{A}^{(1)}(t-)$, $\triangle \mathcal{D}(t) = \mathcal{D}(t) - \mathcal{D}(t-)$.

For the second priority units, we have:

$$(12.2) \quad \mathcal{Q}^{(2)}(t) = \max\left\{0, \mathcal{Q}^{(2)}(t-) + \triangle \mathcal{A}^{(2)}(t) - [\triangle \mathcal{D}(t) - \mathcal{Q}^{(1)}(t-)] 1_{\{\mathcal{Q}^{(1)}(t)=0\}}\right\}.$$

In general, for the $k+1$ st priority units ($k = 1, 2, \dots, \ell-1$) we have

$$(12.3) \quad \begin{aligned} \mathcal{Q}^{(k+1)}(t) = & \max\left\{0, \mathcal{Q}^{(k+1)}(t-) + \triangle \mathcal{A}^{(k+1)}(t) \right. \\ & \left. - \left[\triangle \mathcal{D}(t) - \sum_{i=1}^k \mathcal{Q}^{(i)}(t-) \right] 1_{\{\sum_{i=1}^k \mathcal{Q}^{(i)}(t)=0\}} \right\}. \end{aligned}$$

Equations (12.2) and (12.3) implies that the priority rule is the following. The units are ordered and then leave the system according to their priority as follows. Let, for example, there be 6 units in total, two of them are of highest priority and the rest four are of second priority. Let $C = 3$. Then, after the departure of a group of three units, there will remain only 3 units of the second priority. That is all of the units (i.e. two) of the highest priority and one unit of the second priority leave simultaneously.

For $k = 1, 2, \dots, \ell$, using the notation

$$\mathcal{Q}_k(t) = \mathcal{Q}^{(1)}(t) + \mathcal{Q}^{(2)} + \dots + \mathcal{Q}^{(k)},$$

and

$$\mathcal{A}_k(t) = \mathcal{A}^{(1)}(t) + \mathcal{A}^{(2)} + \dots + \mathcal{A}^{(k)}$$

enables us to write the relations

$$(12.4) \quad \mathcal{Q}_k(t) = \max\{0, \mathcal{Q}_k(t-) + \triangle \mathcal{A}_k(t) - \triangle \mathcal{D}(t)\},$$

which are similar to (12.1). The equivalence of (12.4) and (12.1) - (12.3) is proved by induction in [15].

According to (12.4), the stability condition for this priority queueing system with infinite queues is $\rho_\ell < 1$, where $\rho_\ell = \frac{\lambda}{\mu C}$, λ is the expected number of arrivals per time unit, μ the parameter of departure Poisson process, and C is the aforementioned constant jump of this Poisson process. We will assume that this condition holds for finite buffer models too (for finite buffer models the above stability condition is not required). So, if all the buffers are large, then the losses occur seldom.

In the case of finite buffer models, the representations are similar. Denote the capacity for the total number of units of priority k (the number of units of priority k that can be present simultaneously in the system) by $N^{(k)}$, and $N^{(1)} + N^{(2)} + \dots + N^{(\ell)} = N$. We will assume here that if upon an arrival of a batch, the buffer of the given class is overflowed, then the entire arrival batch is rejected and lost from the system.

Introduce new arrival processes $\overline{\mathcal{A}}^{(k)}(t)$, $k = 1, 2, \dots, \ell$, corresponding to the k th priority as follows. Let the jump $\Delta \overline{\mathcal{A}}^{(k)}(t)$ is defined as

$$(12.5) \quad \Delta \overline{\mathcal{A}}^{(k)}(t) = \Delta \mathcal{A}^{(k)}(t) 1_{\{\mathcal{Q}^{(k)}(t) \leq N^{(k)}\}}.$$

The difference between the process $\mathcal{A}^{(k)}(t)$ and $\overline{\mathcal{A}}^{(k)}(t)$ is only in jumps. In the case of the originally defined process $\mathcal{A}^{(k)}(t)$ the jumps are $\Delta \mathcal{A}^{(k)}(t)$, while in the case of the process $\overline{\mathcal{A}}^{(k)}(t)$ they are $\Delta \overline{\mathcal{A}}^{(k)}(t)$. In addition, it turns out that the process $\overline{\mathcal{A}}^{(k)}(t)$ is not right continuous. There are isolated points at moments of overflowing the buffer of the given priority units.

Then the buffer content for units of the highest priority is defined by equations:

$$(12.6) \quad \mathcal{Q}^{(1)}(t) = \max\{\mathcal{Q}^{(1)}(t-) + \Delta \mathcal{A}^{(1)}(t) - \Delta \mathcal{D}(t)\},$$

$$(12.7) \quad \mathcal{Q}^{(1)}(t+) = \max\{\mathcal{Q}^{(1)}(t-) + \Delta \overline{\mathcal{A}}^{(1)}(t) - \Delta \mathcal{D}(t)\}.$$

In all continuity points of the processes $\mathcal{Q}^{(k)}(t)$, $k = 1, 2, \dots, \ell$, one can obtain general representation similar to that of (12.4). Denoting

$$\overline{\mathcal{A}}_k(t) = \overline{\mathcal{A}}^{(1)}(t) + \overline{\mathcal{A}}^{(2)}(t) + \dots + \overline{\mathcal{A}}^{(k)}(t),$$

one can show the following representations:

$$(12.8) \quad \mathcal{Q}_k(t) = \max\{\mathcal{Q}_k(t-) + \Delta \mathcal{A}_k(t) - \Delta \mathcal{D}(t)\},$$

$$(12.9) \quad \mathcal{Q}_k(t+) = \max\{\mathcal{Q}_k(t-) + \Delta \overline{\mathcal{A}}_k(t) - \Delta \mathcal{D}(t)\},$$

which are supposed to be correct for continuity points of the processes $\mathcal{Q}_k(t)$ only.

According to representations (12.8) and (12.9), the cumulative buffer contents $\mathcal{Q}_k(t)$, $k = 1, 2, \dots, \ell$, in continuity points behave as usual queue-length processes in $GI/M^C/1/N_k$ queueing systems, where $N_k = N^{(1)} + N^{(2)} + \dots + N^{(k)}$ ($N_\ell = N$). However, the behavior of the number of losses is essentially different. The losses in $GI/M^C/1/N_k$ queues are not adequate to the losses in the corresponding cumulative buffers $\mathcal{Q}_k(t)$. Specifically, the losses in $GI/M^C/1/N_k$ queues occur only in the case when the buffer is overflowed upon arrival of a customer meeting all of the waiting places busy. The losses in the cumulative buffers $\mathcal{Q}_k(t)$ can occur in many cases, when one of the buffers of priority units (say the j th buffer, $1 \leq j \leq k$) is overflowed.

However, in some cases when the values $N_1 < N_2 < \dots < N_\ell$ all are large, the correspondence between $GI/M^C/1/N_k$ queues and finite buffers models, giving us useful asymptotic result, is possible. Specifically, the loss probability of a customer arriving to one of the first k buffers is not greater than $\pi_1 + \pi_2 + \dots + \pi_k$, where π_i denotes the loss probability in the corresponding $GI/M^C/1/N_i$ queueing system.

Let $A_i(x)$ ($i = 1, 2, \dots, \ell$) denote the probability distribution function of interarrival time between arrivals of one of the first i priority customers, $\hat{A}_i(s)$ is the Laplace-Stieltjes transform of $A_i(x)$. All of the probabilities π_k are small and decrease geometrically fast (see Theorem 12.1 below). Since the loads $\rho_1, \rho_2, \dots, \rho_\ell$ of corresponding cumulative processes increase, i.e. $\rho_1 < \rho_2 < \dots < \rho_\ell$, then the roots φ_k of the corresponding functional equations appearing in the aforementioned theorem are ordered $\varphi_1 < \varphi_2 < \dots < \varphi_k$. Such type of dependence between the roots φ_k and the loads ρ_k , $k = 1, 2, \dots, \ell$ is because of the special construction of interarrival times. Then, under the assumption that $\varphi_j^{N_j} = o(\varphi_k^{N_k})$, $j < k$, (i.e. the losses of lower priority customers occur much often compared to those of higher priority) the loss probability of the cumulated buffer of the first k priority customers can be approximated by π_k . In this case we have the following theorem.

Theorem 12.1. *For the loss probability π_k of cumulated buffer content we have the following estimation:*

(12.10)

$$\pi_k = \frac{(1 - \rho_k)[1 + C\mu\hat{A}'_k(\mu - \mu\varphi_k^C)]\varphi_k^{N_k}}{(1 - \rho_k)(1 + \varphi_k + \varphi_k^2 + \dots + \varphi_k^{C-1}) - \rho_k[1 + C\mu\hat{A}'_k(\mu - \mu\varphi_k^C)]\varphi_k^{N_k} + o(\varphi_k^{2N_k})},$$

where

$$\begin{aligned}\rho_k &= \frac{\lambda_k}{\mu C}, \\ \lambda_k &= \lambda \sum_{i=1}^k p^{(i)},\end{aligned}$$

and φ_k is the least positive root of the functional equation

$$z = \hat{A}_k(\mu - \mu z^C)$$

in the interval $(0, 1)$.

Proof. We consider the $GI/M^C/1/N_k$ queueing system and, similarly to the proof of Sect. 4, this proof is based on an application of Takács' theorem. Following Miyazawa [59], the loss probability for the $GI/M^Y/1/N_k$ queueing system (in this notation the batch size Y is an integer random variable rather than a deterministic constant that in the cases of the notation $GI/M^C/1/N_k$ considered before) is determined by the formula

$$(12.11) \quad \pi_k = \frac{1}{\sum_{j=0}^{N_k} r_{k,j}},$$

which is similar to relation (4.1). The generating function of $r_{k,j}$ is

$$(12.12) \quad R_k(z) = \sum_{j=0}^{\infty} r_{k,j} z^j = \frac{(1 - Y(z))\hat{A}_k[\mu - \mu Y(z)]}{\hat{A}_k[\mu - \mu Y(z)] - z},$$

where $Y(z)$ is the generating function of a complete batch size. In the case of $GI/M^C/1/N_k$ queueing system $Y(z) = z^C$, and (12.12) is rewritten

$$(12.13) \quad R_k(z) = \frac{(1 - z^C)\hat{A}_k(\mu - \mu z^C)}{\hat{A}_k(\mu - \mu z^C) - z}.$$

Expanding $(1 - z^C)$ in the numerator of (12.13) as $1 - z^C = (1 - z)(1 + z + \dots + z^{C-1})$, we have:

$$(12.14) \quad R_k(z) = \frac{(1 - z)(1 + z + z^2 + \dots + z^{C-1})\hat{A}_k(\mu - \mu z^C)}{\hat{A}_k(\mu - \mu z^C) - z}.$$

Now, let us consider the other generating function given by $\tilde{R}_k(z) = \frac{1}{1-z}R(z)$. From (12.14) we obtain:

$$(12.15) \quad \tilde{R}_k(z) = \sum_{j=0}^{\infty} \tilde{r}_{k,j} z^j = \frac{(1 + z + z^2 + \dots + z^{C-1})\hat{A}_k(\mu - \mu z^C)}{\hat{A}_k(\mu - \mu z^C) - z},$$

and the loss probability is

$$(12.16) \quad \pi_k = \frac{1}{\tilde{r}_{k,N_k}}.$$

Now, application of Takács' theorem is straightforward, because the term

$$\frac{\hat{A}_k(\mu - \mu z^C)}{\hat{A}_k(\mu - \mu z^C) - z}$$

of (12.15) has the representation similar to (1.3), and therefore the corresponding coefficients of the generating function satisfy recurrence relation (1.2).

For large N_k we obtain:

$$(12.17) \quad \begin{aligned} \tilde{r}_{k,N_k} &= \frac{1 + \varphi_k + \dots + \varphi_k^{C-1}}{\varphi_k^{N_k}} \cdot \frac{1}{1 + \mu C \hat{A}(\mu - \mu \varphi_k^C)} \\ &\quad + \frac{(1 + \varphi_k + \dots + \varphi_k^{C-1})\rho_k}{\rho_k - 1} + o(1). \end{aligned}$$

Asymptotic relation (12.10) follows from (12.16) and (12.17). \square

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